CHAPTER 13

Limits and Derivatives

EXAM DRILL

SOLUTIONS

1. **(d) :** We have, $\lim_{x\to 1}\frac{x^2-1}{x^2+1}$

$=\frac{1-1}{1+1}=\frac{0}{2}=0$

2. **(c) :** We have, $\lim_{x\to\frac{\pi}{2}}\frac{1-\cos x}{\sin x}$

$=\lim_{x\to\frac{\pi}{2}}\frac{2\sin^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$

$=\lim_{x\to\frac{\pi}{2}}\tan\left(\frac{x}{2}\right)=\tan\left(\frac{\pi}{4}\right)=1$

3. **(a) :** $\lim_{x\to 3}\frac{e^x-e^3}{x-3}=\lim_{x\to 3}e^3\left(\frac{e^{x-3}-1}{x-3}\right)$

$=\lim_{x-3\to 0}e^3\left(\frac{e^{x-3}-1}{x-3}\right)$ $[\because x\to 3\Rightarrow x-3\to 0]$

$=e^3\times 1=e^3$

4. **(b) :** We have, $f(x)=\cos x-\sin x$

$f'(x)=\frac{d}{dx}(\cos x)-\frac{d}{dx}(\sin x)=-\sin x-\cos x$

$=-(\sin x+\cos x)$

$f'\left(\frac{2\pi}{3}\right)=-\left(\sin\frac{2\pi}{3}+\cos\frac{2\pi}{3}\right)=-\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)$

$\therefore\quad f'\left(\frac{2\pi}{3}\right)=\frac{1-\sqrt{3}}{2}$

5. **(c) :** $\lim_{x\to 1}\left[\frac{x^2+1}{x+100}\right]=\frac{1+1}{1+100}=\frac{2}{101}$

6. **(a) :** Since $\lim_{x\to 0}x=0$ and $-1\le\sin\frac{1}{x}\le 1\ \forall\ x\in R$

$\therefore\quad\lim_{x\to 0}x\sin\frac{1}{x}=0$

7. **(b) :** We have, $f(x)=\frac{x^3+x^2+1}{x}$

$\Rightarrow\quad f(x)=x^2+x+\frac{1}{x}\quad\therefore\ f'(x)=2x+1-\frac{1}{x^2}$

8. **(a) :** We have, $y=\frac{\sin(x+9)}{\cos x}$

Now, $\frac{dy}{dx}=\frac{\cos x\cos(x+9)-\sin(x+9)(-\sin x)}{(\cos x)^2}$

$=\frac{\cos x\cos(x+9)+\sin x\sin(x+9)}{\cos^2 x}$

Hence, $\left(\frac{dy}{dx}\right)_{x=0}=\frac{\cos 9}{1}=\cos 9$

9. **(b) :** We have, $\lim_{x\to 1}\frac{x^m-1}{x^n-1}$

$=\lim_{x\to 1}\frac{\frac{x^m-1}{x-1}}{\frac{x^n-1}{x-1}}=\frac{\lim_{x\to 1}\frac{x^m-1^m}{x-1}}{\lim_{x\to 1}\frac{x^n-1^n}{x-1}}=\frac{m(1)^{m-1}}{n(1)^{n-1}}=\frac{m}{n}$

10. **(b) :** Given, $f(x)=1-x+x^2-x^3+\ldots-x^{99}+x^{100}$

$\therefore\quad f'(x)=0-1+2x-3x^2+\ldots-99x^{98}+100x^{99}$

So, $f'(1)=-1+2(1)-3(1)^2+\ldots-99(1)^{98}+100(1)^{99}$

$=-1+2-3+\ldots-99+100$

$=(2+4+\ldots+98+100)-(1+3+5+\ldots+99)$

$=\frac{50}{2}[2\times 2+(50-1)2]-\frac{50}{2}[2\times 1+(50-1)2]$

$=25[4+49\times 2]-25[2+49\times 2]$

$=2550-2500=50$

11. We have, $f(x)=x^4-3x^3+2x^2-x+1$

$\therefore\quad f'(x)=4x^3-3\times 3x^2+2\times 2x-1+0$

$=4x^3-9x^2+4x-1$

So, $f'(1)=4(1)^3-9(1)^2+4(1)-1$

$=4-9+4-1=-2$

12. We have, $\lim_{x\to 0}\frac{e^{-x}-1}{x}$

Putting $-x=y$, we get

$\lim_{y\to 0}\frac{e^y-1}{-y}=-\lim_{y\to 0}\frac{e^y-1}{y}=-1$

13. We have, $\lim_{x\to 4}\frac{x^2-16}{x-4}=\lim_{x\to 4}\frac{x^2-(4)^2}{x-4}$

$=\lim_{x\to 4}\frac{(x+4)(x-4)}{(x-4)}=\lim_{x\to 4}(x+4)=8$

14. We have, $f(x)=\sin x\cos^3 x$

$\therefore\quad f'(x)=\sin x(-\sin x)\cdot 3\cos^2(x)+\cos^3 x(\cos x)$

$=\cos^2 x(\cos^2 x-3\sin^2 x)$

15. We have, $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$

$=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}=2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2=2$

16. We have, $f(x) = (3x + 5)(1 + \tan x)$

$$\frac{df(x)}{dx} = \frac{d}{dx}[(3x+5)(1+\tan x)]$$

$$= (3x+5)\frac{d}{dx}(1+\tan x) + (1+\tan x)\frac{d}{dx}(3x+5)$$

$= (3x + 5)(\sec^2 x) + (1 + \tan x)(3)$

$= 3x\sec^2 x + 5\sec^2 x + 3 + 3\tan x$

$= 3(1 + \tan x + x\sec^2 x) + 5\sec^2 x$

17. We have, $\lim\limits_{x\to 2}\dfrac{\frac{1}{x}-\frac{1}{2}}{x-2} = \lim\limits_{x\to 2}\dfrac{\frac{2-x}{2x}}{(x-2)} = \lim\limits_{x\to 2}\dfrac{-1}{2x} = \dfrac{-1}{4}$

18. We have, $\lim\limits_{x\to 0}\dfrac{\cos ax}{\cos bx} = \dfrac{1}{1} = 1$

19. Let $x - 2 = t$, As $x \to 2$, $t \to 0$

$$\lim_{x\to 2}\frac{e^x - e^2}{x-2} = \lim_{t\to 0}\frac{e^{t+2}-e^2}{t} = \lim_{t\to 0} e^2\cdot\frac{e^t - 1}{t}$$

$= e^2 \times 1 = e^2$

20. We have, $y = \dfrac{x^3\sin x}{\cos x} = x^3\tan x$

$\therefore \dfrac{dy}{dx} = 3x^2\tan x + x^3\sec^2 x$

$= x^2(3\tan x + x\sec^2 x)$

21. (i)(c) : $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(mx^2+n) = \lim\limits_{h\to 0} m(1-h)^2 + n$

$= m + n$

(ii)(a) : $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1+}(nx^3 - m) = \lim\limits_{h\to 0} n(1-h)^3 - m$

$= n - m$

(iii) (a) : Since, the limit exists at $x = 1$

$\Rightarrow$ L.H.L. $= f(1) =$ R.H.L.

$\Rightarrow m + n = 2 = n - m$

By solving, we get $m = 0$

$n = 2$

(iv) (b) : $m \cdot n = 0 \times 2 = 0$

(v) (c) : Constant term of $f(x)$

22. (i) Let $y = \dfrac{x^2\cos\frac{\pi}{4}}{\sin x} = \dfrac{\frac{x^2}{\sqrt{2}}}{\sin x} = \dfrac{1}{\sqrt{2}}\cdot\dfrac{x^2}{\sin x}$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{2}}\left[\frac{(\sin x)\cdot\frac{d}{dx}(x^2) - (x^2)\frac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{\sin x\cdot 2x - x^2\cos x}{\sin^2 x}\right]$$

$$= \frac{x}{\sqrt{2}}[2\operatorname{cosec} x - x\cot x\operatorname{cosec} x]$$

$$= \frac{x}{\sqrt{2}}\operatorname{cosec} x\,[2 - x\cot x]$$

(ii) Let $y = x^3(3\sin x - 5x\cos x)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(3x^3\sin x) - \frac{d}{dx}(5x^4\cos x)$$

$$= 3\left\{\frac{d}{dx}(x^3)\cdot\sin x + x^3\frac{d}{dx}(\sin x)\right\}$$

$$-5\left\{\cos x\frac{d}{dx}(x^4) + x^4\frac{d}{dx}(\cos x)\right\}$$

$= 3(3x^2\sin x + x^3\cos x) - 5(4x^3\cos x - x^4\sin x)$

$$\frac{dy}{dx} = 5x^4\sin x - 17x^3\cos x + 9x^2\sin x$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 0$$

23. We have, $\lim\limits_{x\to 0}(1+\sin x)^{2\cot x}$

$$= e^{\lim\limits_{x\to 0}\sin x\times 2\cot x} = e^{\lim\limits_{x\to 0} 2\cos x} = e^2$$

24. We have, $\lim\limits_{x\to\pi}\dfrac{1+\cos^3 x}{\sin^2 x}$

$$= \lim_{x\to\pi}\frac{(1+\cos x)(1-\cos x+\cos^2 x)}{(1-\cos x)(1+\cos x)}$$

$$= \lim_{x\to\pi}\frac{1-\cos x+\cos^2 x}{1-\cos x} = \frac{1+1+1}{1+1} = \frac{3}{2}$$

25. Let $y = \dfrac{x-1}{x^2}$

$$\frac{dy}{dx} = \frac{(x^2)\cdot\frac{d}{dx}(x-1) - (x-1)\cdot\frac{d}{dx}(x^2)}{(x^2)^2}$$

[Using Quotient rule]

$$= \frac{x^2\cdot 1 - (x-1)\cdot 2x}{x^4} = \frac{x^2 - 2x^2 + 2x}{x^4}$$

$$= \frac{-x^2 + 2x}{x^4} = \frac{x(2-x)}{x^4}$$

$$\therefore \frac{dy}{dx} = \frac{2-x}{x^3}$$

OR

Let $y = \sin x\,(x^2 + 5x + 7)$

$$\therefore \frac{dy}{dx} = (x^2+5x+7)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2+5x+7)$$

$= (x^2 + 5x + 7)\cos x + \sin x\,(2x + 5)$

26. We have given,

$$\lim_{x\to 1}\frac{x^4 - \sqrt{x}}{\sqrt{x}-1} = \lim_{x\to 1}\frac{\sqrt{x}[(x)^{7/2}-1]}{\sqrt{x}-1}$$

$$= \frac{\lim\limits_{x\to 1}\frac{x^{7/2}-1}{x-1}\cdot\sqrt{x}}{\lim\limits_{x\to 1}\frac{(x)^{1/2}-1}{x-1}} = \frac{\frac{7}{2}(1)^{\frac{7}{2}-1}}{\frac{1}{2}(1)^{\frac{1}{2}-1}} = \frac{\frac{7}{2}}{\frac{1}{2}} = 7$$

27. We have, $\lim_{x\to 0}\dfrac{\sin^2 2x}{\sin^2 4x}$

$=\lim_{x\to 0}\dfrac{\sin^2 2x}{[\sin 2(2x)]^2}$

$=\lim_{x\to 0}\dfrac{\sin^2 2x}{(2\sin 2x\cos 2x)^2}$

$=\lim_{x\to 0}\dfrac{\sin^2 2x}{4\sin^2 2x\cos^2 2x}=\lim_{x\to 0}\dfrac{1}{4\cos^2 2x}=\dfrac{1}{4}$

28. Let $y=\left(x^2+\dfrac{1}{x^2}\right)^3$

$=(x^2)^3+\left(\dfrac{1}{x^2}\right)^3+3x^4\cdot\dfrac{1}{x^2}+3\cdot x^2\cdot\dfrac{1}{x^4}$

$= x^6 + x^{-6} + 3x^2 + 3x^{-2}$

Now, $\dfrac{dy}{dx} = 6x^5 + (-6)x^{-7} + 3\cdot 2\cdot x + 3(-2)x^{-3}$

$= 6x^5 - 6x^{-7} + 6x - 6x^{-3}$

$= 6(x^5 - x^{-7} + x - x^{-3})$

29. We have, $\lim_{x\to 0}\dfrac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}$

$=\lim_{x\to 0}\dfrac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}\times\dfrac{\sqrt{1+x^3}+\sqrt{1-x^3}}{\sqrt{1+x^3}+\sqrt{1-x^3}}$

$=\lim_{x\to 0}\dfrac{(1+x^3)-(1-x^3)}{x^2(\sqrt{1+x^3}+\sqrt{1-x^3})}$

$=\lim_{x\to 0}\dfrac{1+x^3-1+x^3}{x^2(\sqrt{1+x^3}+\sqrt{1-x^3})}$

$=\lim_{x\to 0}\dfrac{2x^3}{x^2(\sqrt{1+x^3}+\sqrt{1-x^3})}$

$=\lim_{x\to 0}\dfrac{2x}{(\sqrt{1+x^3}+\sqrt{1-x^3})}=\dfrac{0}{2}=0$

30. We have, $y=\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}=\sqrt{\dfrac{2\sin^2 x}{2\cos^2 x}}=\sqrt{\tan^2 x}$

$\Rightarrow\quad y=|\tan x|$, where $x\in\left(0,\dfrac{\pi}{2}\right)\cup\left(\dfrac{\pi}{2},\pi\right)$

$$\Rightarrow\quad y=\begin{cases}\tan x, & x\in\left(0,\dfrac{\pi}{2}\right)\\ -\tan x, & x\in\left(\dfrac{\pi}{2},\pi\right)\end{cases}$$

$$\therefore\quad \frac{dy}{dx}=\begin{cases}\sec^2 x, & \text{if } x\in\left(0,\dfrac{\pi}{2}\right)\\ -\sec^2 x, & \text{if } x\in\left(\dfrac{\pi}{2},\pi\right)\end{cases}$$

31. We have, $\lim_{x\to 0}\dfrac{x2^x-x}{1-\cos x}=\lim_{x\to 0}\dfrac{x(2^x-1)}{1-\cos x}$

$=\lim_{x\to 0}\dfrac{\dfrac{x^2(2^x-1)}{x}}{2\sin^2\dfrac{x}{2}}=\lim_{x\to 0}\dfrac{x^2}{2\left(\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\cdot\dfrac{x^2}{4}}\cdot\dfrac{2^x-1}{x}$

$= 2\cdot\log 2$ $\quad\left[\because \lim_{x\to 0}\dfrac{a^x-1}{x}=\log_e a\right]$

$= \log(2^2) = \log 4$

OR

$\lim_{x\to 0}\dfrac{\log(6+x)-\log(6-x)}{x}$

$=\lim_{x\to 0}\dfrac{\log 6\left(1+\frac{x}{6}\right)-\log 6\left(1-\frac{x}{6}\right)}{x}$

$=\lim_{x\to 0}\dfrac{\left[\log 6+\log\left(1+\frac{x}{6}\right)\right]-\left[\log 6+\log\left(1-\frac{x}{6}\right)\right]}{x}$

$=\lim_{x\to 0}\left[\dfrac{\log\left(1+\frac{x}{6}\right)}{x}-\dfrac{\log\left(1-\frac{x}{6}\right)}{x}\right]$

$=\lim_{x\to 0}\dfrac{1}{6}\dfrac{\log\left(1+\frac{x}{6}\right)}{\frac{x}{6}}+\lim_{x\to 0}\dfrac{1}{6}\dfrac{\log\left(1-\frac{x}{6}\right)}{\frac{-x}{6}}$

$\left[\because\ \lim_{x\to 0}\dfrac{\log(1+x)}{x}=\lim_{x\to 0}\dfrac{\log(1-x)}{-x}=1\right]$

$=\dfrac{1}{6}\times 1+\dfrac{1}{6}\times 1=\dfrac{1}{6}+\dfrac{1}{6}=\dfrac{2}{6}=\dfrac{1}{3}$

32. We have, $\lim_{x\to y}\dfrac{\tan x-\tan y}{x-y}$

Here, independent variable x is not tending to zero rather x is tending to y, hence put $x = y + h$. Let, $x = y + h$, then as $x \to y$, $h \to 0$

Now, $\lim_{x\to y}\dfrac{\tan x-\tan y}{x-y}$

$=\lim_{h\to 0}\dfrac{\tan(y+h)-\tan y}{y+h-y}$

$=\lim_{h\to 0}\dfrac{1}{h}\left[\dfrac{\sin(y+h)}{\cos(y+h)}-\dfrac{\sin y}{\cos y}\right]$

$=\lim_{h\to 0}\dfrac{\sin(y+h)\cos y-\sin y\cos(y+h)}{h\cos y\cos(y+h)}$

$=\lim_{h\to 0}\dfrac{\sin(y+h-y)}{h\cos(y+h)\cos y}$

$=\lim_{h\to 0}\dfrac{\sin h}{h}\dfrac{1}{\cos(y+h)\cos y}$

$=1\cdot\dfrac{1}{\cos^2 y}=\sec^2 y$

OR

Let $y = (x \sin x + \cos x)(x \cos x - \sin x)$

$$\therefore \quad \frac{dy}{dx} = (x\cos x - \sin x)\frac{d}{dx}(x\sin x + \cos x) + (x\sin x + \cos x)\frac{d}{dx}(x\cos x - \sin x)$$

$= (x \cos x - \sin x)(\sin x + x \cos x - \sin x) + (x \sin x + \cos x) \times (\cos x - x \sin x - \cos x)$

$= (x \cos x - \sin x)x \cos x + (x \sin x + \cos x)(-x \sin x)$

$= x^2 \cos^2 x - x \sin x \cos x - x^2 \sin^2 x - x \sin x \cos x$

$= x^2 (\cos^2 x - \sin^2 x) - 2x \sin x \cos x$

$= x^2 \cos 2x - x \sin 2x$

33. We have, $\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}$

$$= \lim_{x\to 0}\frac{(1-\cos x\sqrt{\cos 2x})(1+\cos x\sqrt{\cos 2x})}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x\cos 2x}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\frac{1-(1-\sin^2 x)(1-2\sin^2 x)}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\frac{1-(1-3\sin^2 x+2\sin^4 x)}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\frac{\sin^2 x(3-2\sin^2 x)}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\left(\frac{3-2\sin^2 x}{1+\cos x\sqrt{\cos 2x}}\right)$$

$$= 1^2\left(\frac{3}{1+1}\right) = \frac{3}{2}$$

34. Let $y = f(x) = \frac{\sin x}{x}$

Using first principle, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{\sin(x+h)}{x+h}-\frac{\sin x}{x}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\frac{x\sin(x+h)-x\sin x-h\sin x}{x(x+h)}$$

$$= \lim_{h\to 0}\frac{1}{h}\frac{[\sin(x+h)-\sin x]}{x+h} - \lim_{h\to 0}\frac{\sin x}{x(x+h)}$$

$$= \lim_{h\to 0}\frac{1}{h}\frac{2}{x+h}\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2} - \frac{\sin x}{x^2}$$

$$= \lim_{h\to 0}\frac{2}{h(x+h)}\cos\left(\frac{2x+h}{2}\right)\cdot\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\cdot\frac{h}{2} - \frac{\sin x}{x^2}$$

$$= \frac{1}{x}\cos x - \frac{\sin x}{x^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$$

35. Let $f(x) = \sqrt{\sin x}$

Using first principle, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{\sin(x+h)}-\sqrt{\sin x}}{h}$$

$$= \lim_{h\to 0}\frac{(\sqrt{\sin(x+h)}-\sqrt{\sin x})(\sqrt{\sin(x+h)}+\sqrt{\sin x})}{h(\sqrt{\sin(x+h)}+\sqrt{\sin x})}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h\{\sqrt{\sin(x+h)}+\sqrt{\sin x}\}}$$

$$= \lim_{h\to 0}\frac{2\sin\left\{\frac{h}{2}\right\}\cos\left\{\frac{2x+h}{2}\right\}}{h\{\sqrt{\sin(x+h)}+\sqrt{\sin x}\}}$$

$$\left[\because \sin C - \sin D = 2\sin\frac{C-D}{2}\cos\frac{C+D}{2}\right]$$

$$= \lim_{h\to 0}\frac{\sin h/2}{h/2}\times\lim_{h\to 0}\frac{\cos(x+h/2)}{\{\sqrt{\sin(x+h)}+\sqrt{\sin x}\}}$$

$$= \frac{\cos x}{\sqrt{\sin x}+\sqrt{\sin x}} = \frac{\cos x}{2\sqrt{\sin x}}$$

OR

We have, $y = \frac{x}{x+5}$ ...(i)

On differentiating both sides of eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x+5}\right)$$

$$= \frac{(x+5)\frac{d}{dx}(x)-(x)\frac{d}{dx}(x+5)}{(x+5)^2} \quad \text{[By Quotient rule]}$$

$$= \frac{(x+5)(1)-x(1+0)}{(x+5)^2} = \frac{x+5-x}{(x+5)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{5}{(x+5)^2}$$

Now, L.H.S. $= x\frac{dy}{dx} = \frac{5x}{(x+5)^2}$ ...(ii)

and R.H.S. $= y(1-y) = \frac{x}{x+5}\left(1-\frac{x}{x+5}\right)$

$$= \frac{x}{x+5}\left(\frac{x+5-x}{x+5}\right)$$

$$= \frac{5x}{(x+5)^2} \quad \text{...(iii)}$$

From eqs. (ii) and (iii), we get

$$x\frac{dy}{dx} = y = (1-y)$$

Hence proved.