Limits and Derivatives

CHAPTER 13

NCERT FOCUS

SOLUTIONS

EXERCISE - 13.1

1. We have, $\lim_{x\to 3}(x+3)=3+3=6.$

2. We have, $\lim_{x\to\pi}\left(x-\frac{22}{7}\right)=\pi-\frac{22}{7}.$

3. We have, $\lim_{r\to 1}(\pi r^2)=\pi\cdot(1)^2=\pi.$

4. We have, $\lim_{x\to 4}\frac{4x+3}{x-2}=\frac{4(4)+3}{4-2}=\frac{19}{2}.$

5. We have, $\lim_{x\to -1}\frac{x^{10}+x^5+1}{x-1}$

$=\frac{(-1)^{10}+(-1)^5+1}{-1-1}=\frac{1-1+1}{-2}=\frac{-1}{2}.$

6. We have, $\lim_{x\to 0}\frac{(x+1)^5-1}{x}$ $\left(\frac{0}{0}\text{ form}\right)$

Put $y=1+x$, so that $y\to 1$ as $x\to 0$

Then $\lim_{x\to 0}\frac{(x+1)^5-1}{x}=\lim_{y\to 1}\frac{y^5-1}{y-1}=5(1)^{5-1}=5.$

[Using $\lim_{x\to a}\frac{x^n-a^n}{x-a}=n(a)^{n-1}$]

7. We have, $\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}$ $\left(\frac{0}{0}\text{ form}\right)$

$=\lim_{x\to 2}\frac{[3x^2-6x+5x-10]}{(x-2)(x+2)}$

$=\lim_{x\to 2}\frac{[3x(x-2)+5(x-2)]}{(x-2)(x+2)}$

$=\lim_{x\to 2}\frac{(3x+5)(x-2)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{3x+5}{x+2}$

$=\frac{3(2)+5}{2+2}=\frac{6+5}{4}=\frac{11}{4}.$

8. We have, $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$ $\left(\frac{0}{0}\text{ form}\right)$

$=\lim_{x\to 3}\frac{(x^2-9)(x^2+9)}{2x^2-6x+x-3}$

$=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{[2x(x-3)+1(x-3)]}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(2x+1)(x-3)}$

$=\frac{(3+3)(9+9)}{(6+1)}=\frac{6\times 18}{7}=\frac{108}{7}.$

9. We have, $\lim_{x\to 0}\frac{ax+b}{cx+1}=\frac{a\cdot(0)+b}{c\cdot(0)+1}=b.$

10. We have, $\lim_{z\to 1}\frac{z^{1/3}-1}{z^{1/6}-1}$ $\left(\frac{0}{0}\text{ form}\right)$

$=\lim_{z\to 1}\frac{(z^{1/6})^2-1}{(z^{1/6}-1)}=\lim_{z\to 1}\frac{(z^{1/6}-1)(z^{1/6}+1)}{(z^{1/6}-1)}$

$=\lim_{z\to 1}(z^{1/6}+1)=1^{1/6}+1=1+1=2$

11. We have, $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}$

$=\frac{a\cdot(1)^2+b\cdot(1)+c}{c\cdot(1)^2+b\cdot(1)+a}=\frac{a+b+c}{c+b+a}=1.$

12. We have, $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

$=\lim_{x\to -2}\frac{(x+2)/2x}{x+2}=\lim_{x\to -2}\frac{1}{2x}=\frac{1}{2(-2)}=\frac{1}{-4}.$

13. We have, $\lim_{x\to 0}\frac{\sin ax}{bx}=\lim_{x\to 0}\frac{\sin ax}{ax}\cdot\frac{ax}{bx}$

$=\lim_{x\to 0}\frac{\sin ax}{ax}\cdot\frac{a}{b}=\frac{a}{b}\left(\lim_{x\to 0}\frac{\sin ax}{ax}\right)=\frac{a}{b}$

$\left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$

14. We have, $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$, $a, b\neq 0$

$=\lim_{x\to 0}\frac{\sin ax}{ax}\cdot\frac{bx}{\sin bx}\cdot\frac{ax}{bx}$

$=\frac{a}{b}\cdot\lim_{x\to 0}\frac{\sin ax}{ax}\cdot\lim_{x\to 0}\frac{1}{\left(\frac{\sin bx}{bx}\right)}$

$=\frac{a}{b}\cdot(1)\cdot(1)=\frac{a}{b}$ $\left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$

15. We have, $\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$ ... (i)

As $x\to\pi \Rightarrow \pi-x\to 0$

From (i), we have

$\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{(\pi-x)}\cdot\frac{1}{\pi}=\frac{1}{\pi}\cdot 1=\frac{1}{\pi}$ $\left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$

16. We have, $\lim_{x\to 0}\frac{\cos x}{\pi-x}=\frac{\cos 0}{\pi-0}=\frac{1}{\pi}$

17. We have, $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

$=\lim_{x\to 0}\frac{\cos 2x-1}{-(1-\cos x)}$

$=\lim_{x\to 0}\frac{(1-\cos 2x)}{(1-\cos x)}\times\frac{(1+\cos x)}{(1+\cos x)}$

$= \lim_{x \to 0} \frac{(2\sin^2 x)(1+\cos x)}{1-\cos^2 x} = \lim_{x \to 0} 2(1+\cos x)$

$= 2(1+\cos 0) = 2(1+1) = 2 \times 2 = 4$

18. We have, $\lim_{x \to 0} \frac{ax + x\cos x}{b\sin x}$

$= \lim_{x \to 0} \frac{x(a+\cos x)}{b\sin x} = \lim_{x \to 0} \frac{x}{\sin x} \frac{(a+\cos x)}{b}$

$= \left(\lim_{x \to 0} \frac{x}{\sin x}\right) \cdot \left(\lim_{x \to 0} \frac{a+\cos x}{b}\right)$

$= (1) \cdot \frac{a+\cos 0}{b} = \frac{a+1}{b}$

19. We have, $\lim_{x \to 0} x\sec x = 0 \cdot 1 = 0$

20. We have, $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$

Dividing the numerator and denominator by x, we get

$\lim_{x \to 0} \frac{\frac{\sin ax}{x} + b}{a + \frac{\sin bx}{x}} = \lim_{x \to 0} \frac{\frac{\sin ax}{ax} \cdot a + b}{a + \frac{\sin bx}{bx} \cdot b}$

$= \frac{a\left[\lim_{x \to 0} \frac{\sin ax}{ax}\right] + b}{a + \left[\lim_{x \to 0} \frac{\sin bx}{bx}\right] \cdot b} = \frac{a(1)+b}{a+b \cdot (1)} \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$

$= \frac{a+b}{a+b} = 1$

21. We have, $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$

$= \lim_{x \to 0} \left[\frac{1-\cos x}{\sin x}\right] = \lim_{x \to 0} \frac{1-1+2\sin^2(x/2)}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$= \lim_{x \to 0} \frac{\sin x/2}{\cos x/2} = \lim_{x \to 0} \tan(x/2) = 0$

22. We have, $\lim_{x \to \frac{\pi}{2}} \left(\frac{\tan 2x}{x - \frac{\pi}{2}}\right) \quad \left(\frac{0}{0} \text{ form}\right)$

Put $x = \frac{\pi}{2} + y$; if $x \to \frac{\pi}{2} \Rightarrow y \to 0$

$\lim_{y \to 0} \frac{\tan 2\left(\frac{\pi}{2}+y\right)}{\frac{\pi}{2}+y-\frac{\pi}{2}} = \lim_{y \to 0} \frac{\tan(\pi+2y)}{y} = \left(\lim_{y \to 0} \frac{\tan 2y}{2y}\right) \times 2 = 2$

23. We have, $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$

For $\lim_{x \to 0} f(x)$, we have to evaluate $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ separately as $f(x)$ is defined differently on left and right of 0.

Now, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x+3) = 2 \cdot (0) + 3 = 3$

and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 3(x+1) = 3(0+1) = 3$

Thus we see, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 3$

$\therefore \lim_{x \to 0} f(x) = 3$

Now, $\lim_{x \to 1} f(x) = \lim_{x \to 1} 3(x+1) = 3 \times 2 = 6$

$\therefore \lim_{x \to 1} f(x) = 6$ and $\lim_{x \to 0} f(x) = 3$

24. We have, $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$

Now, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 - 1) = 1^2 - 1 = 0$

and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-x^2 - 1) = -(1)^2 - 1 = -1 - 1 = -2$

$\because \lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$

$\therefore \lim_{x \to 1} f(x)$ does not exist.

25. We have, $f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

Now, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \left(\frac{-x}{x}\right)$

$[\because |x| = -x \text{ for } x < 0]$

$= \lim_{x \to 0^-} (-1) = -1$

and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \left(\frac{x}{x}\right)$

$[\because |x| = x, \text{ for } x > 0]$

$= \lim_{x \to 0^+} (1) = 1$

$\because \lim_{x \to 0^-} f(x) \ne \lim_{x \to 0^+} f(x)$

Thus $\lim_{x \to 0} f(x)$ does not exist.

26. We have, $f(x) = \begin{cases} \frac{x}{|x|}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

Now, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} \left(\frac{x}{-x}\right)$

$[\because |x| = -x, \text{ for } x < 0]$

$= \lim_{x \to 0^-} (-1) = -1$

and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{x}$

$[\because |x| = x, \text{ for } x > 0]$

$= \lim_{x \to 0^+} (1) = 1$

Thus, $\lim_{x \to 0^-} f(x) \ne \lim_{x \to 0^+} f(x)$

$\therefore \lim_{x \to 0} f(x)$ does not exist.

27. We have, $f(x) = |x| - 5$

Now, $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (|x| - 5) = \lim_{x \to 5^-} (x - 5)$

[$\because$ x is slightly less than 5. $\therefore$ $|x| = x$, for $x > 0$]

$= 5 - 5 = 0$

and $\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (|x| - 5)$

$= \lim_{x \to 5^+} (x - 5) = 5 - 5 = 0$ [$\because$ $|x| = x$, for $x > 0$]

$\therefore \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = 0$

Thus $\lim_{x \to 5} f(x) = 0$.

28. We have, $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$

and $\lim_{x \to 1} f(x) = f(1)$ ... (i)

So, first we have to find $\lim_{x \to 1} f(x)$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (a + bx) = a + b$

and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (b - ax) = b - a$

Also, $f(1) = 4$

By (i), $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

$\Rightarrow \quad a + b = 4$ ... (ii)

and $b - a = 4$ ... (iii)

Adding (ii) and (iii), we get $2b = 8 \Rightarrow b = 4$

Substituting the value of b in (iii), we get

$4 - a = 4 \Rightarrow a = 0$

Thus $a = 0$ and $b = 4$.

29. We have $f(x) = (x - a_1)(x - a_2) \ldots\ldots (x - a_n)$

$\therefore \lim_{x \to a_1} f(x) = f(a_1) = (a_1 - a_1)(a_1 - a_2) \ldots\ldots (a_1 - a_n)$

$= 0$

Also, $\lim_{x \to a} f(x) = \lim_{x \to a} [(x - a_1)(x - a_2) \ldots (x - a_n)]$

$= \left[\lim_{x \to a} (x - a_1)\right]\left[\lim_{x \to a} (x - a_2)\right] \ldots \left[\lim_{x \to a} (x - a_n)\right]$

$= (a - a_1)(a - a_2) \ldots (a - a_n)$

30. We have $f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$

Case (i) : When $a < 0$, we have

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} (|x| + 1) = \lim_{x \to a^-} (-x + 1) = -a + 1$

Case (ii) : When $a > 0$, we have

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} (|x| - 1) = \lim_{x \to a^+} (x - 1) = a - 1$

Case (iii) : When $a = 0$, we have

$\lim_{x \to a^+} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (|x| - 1) = 0 - 1 = -1$

$\lim_{x \to a^-} f(x) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (|x| + 1) = -0 + 1 = 1$

$\therefore \lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x) \neq f(0)$ [$\because$ $f(0) = 0$]

$\lim_{x \to a} f(x)$ does not exist when $a = 0$.

$\therefore \lim_{x \to a} f(x)$ exists for all $a \neq 0$.

31. We have, $\lim_{x \to 1} \dfrac{f(x) - 2}{x^2 - 1} = \pi$

$\because \lim_{x \to 1} (x^2 - 1) = 0$

$\therefore$ For $\lim_{x \to 1} \dfrac{f(x) - 2}{x^2 - 1}$ to exist, we must have

$\lim_{x \to 1} [f(x) - 2] = 0$

[$\because$ If $\lim_{x \to 1} (f(x) - 2) \neq 0$, then the given limit can't exist]

$\Rightarrow \lim_{x \to 1} f(x) - 2 = 0 \quad \Rightarrow \lim_{x \to 1} f(x) = 2.$

32. We have, $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1 \\ nx^3 + m, & x > 1 \end{cases}$

Now $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (mx^2 + n) = m(0) + n = n$

and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (nx + m) = n(0) + m = m$

But for $\lim_{x \to 0} f(x)$ to exist, we must have

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$ *i.e.*, $n = m$

Hence, $\lim_{x \to 0} f(x)$ exists only if $n = m$.

Now, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (nx + m) = n + m$

and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (nx^3 + m) = n + m$

$\therefore$ Above condition shows that

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = m + n$

Thus, $\lim_{x \to 1} f(x)$ exists $\forall\, n, m$.

EXERCISE - 13.2

1. Let $f(x) = x^2 - 2$

Differentiate with respect to x, we get

$f'(x) = 2x$

At $x = 10$, $f'(10) = 2(10) = 20$

2. Let $f(x) = 99x$

Differentiate with respect to x, we get

$f'(x) = 99$

At $x = 100$, $f'(100) = 99$

3. Let $f(x) = x$

Differentiate with respect to x, we get

$f'(x) = 1$

At $x = 1$, $f'(1) = 1$.

4. (i) Let $f(x) = x^3 - 27$

We have, $\dfrac{d}{dx}(f(x)) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{[(x + h)^3 - 27] - [x^3 - 27]}{h}$

$$= \lim_{h \to 0}\left[\frac{x^3 + 3x^2h + 3xh^2 + h^3 - 27 - x^3 + 27}{h}\right]$$

$$= \lim_{h \to 0}\frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} h\left[\frac{3x^2 + 3xh + h^2}{h}\right] = \lim_{h \to 0}(3x^2 + 3xh + h^2)$$

$= 3x^2 + 3x\,(0) + (0)^2 = 3x^2$

(ii) Let $f(x) = (x - 1)\,(x - 2)$

We have, $\frac{d}{dx}(f(x)) = \lim_{h \to 0}\frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0}\frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h}$$

$$= \lim_{h\to 0}\frac{[(x+h)^2 - 2(x+h) - (x+h) + 2] - [x^2 - 2x - x + 2]}{h}$$

$$= \lim_{h\to 0}\frac{[x^2 + h^2 + 2xh - 3(x+h) + 2] - [x^2 - 3x + 2]}{h}$$

$$= \lim_{h\to 0}\frac{h^2 + 2xh - 3h}{h}$$

$$= \lim_{h\to 0}(h + 2x - 3) = 2x - 3$$

(iii) Let $f(x) = \frac{1}{x^2}$

We have, $\frac{d}{dx}(f(x)) = \lim_{h \to 0}\frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0}\left[\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}\right] = \lim_{h\to 0}\left[\frac{x^2 - (x+h)^2}{(x+h)^2x^2h}\right]$$

$$= \lim_{h \to 0}\left[\frac{x^2 - x^2 - h^2 - 2xh}{(x+h)^2\cdot x^2\cdot h}\right]$$

$$= \lim_{h \to 0}\left[\frac{-h^2 - 2xh}{(x+h)^2\cdot x^2\cdot h}\right]$$

$$= \lim_{h \to 0}\left(\frac{-h}{h}\right)\left[\frac{h + 2x}{(x+h)^2\cdot x^2}\right]$$

$$= -\lim_{h \to 0}\left(\frac{h + 2x}{(x+h)^2\cdot x^2}\right) = \frac{-2x}{x^2\cdot x^2} = \frac{-2}{x^3}$$

(iv) Let $f(x) = \frac{x+1}{x-1}$

We have, $\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0}\left[\frac{\frac{(x+h+1)}{(x+h-1)} - \frac{(x+1)}{(x-1)}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{(x+h+1)(x-1) - (x+1)\,(x+h-1)}{h(x+h-1)(x-1)}\right]$$

$$= \lim_{h \to 0}\left[\frac{x^2 + hx + x - x - h - 1 - [x^2 + hx - x + x + h - 1]}{h(x+h-1)(x-1)}\right]$$

$$= \lim_{h \to 0}\left[\frac{(x^2 + hx - h - 1) - (x^2 + hx + h - 1)}{h(x+h-1)(x-1)}\right]$$

$$= \lim_{h \to 0}\left[\frac{x^2 + hx - h - 1 - x^2 - hx - h + 1}{h(x+h-1)(x-1)}\right]$$

$$= \lim_{h \to 0}\left[\frac{-2h}{h(x+h-1)\,(x-1)}\right]$$

$$= \lim_{h \to 0}\left[\frac{-2}{(x+h-1)(x-1)}\right]$$

$$= \frac{-2}{(x+0-1)(x-1)} = \frac{-2}{(x-1)^2}$$

5. We have, $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + ... + \frac{x^2}{2} + x + 1$ ... (i)

Differentiating (i) with respect to x, we get

$$f'(x) = \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + ... + \frac{2x}{2} + 1$$

$\Rightarrow\ f'(x) = x^{99} + x^{98} + ... + x + 1$

At $x = 1, f'(1) = 1^{99} + 1^{98} + ... + 1 + 1 = 100$

and $f'(0) = 0 + 0 + ... + 0 + 1 = 1$

Hence $f'(1) = 100\,f'(0)$.

6. Let $f(x) = x^n + ax^{n-1} + ... + a^{n-1}x + a^n$ ... (i)

Differentiating (i) with respect to x, we get

$f'(x) = nx^{n-1} + (n - 1)\,ax^{n-2} + ... + a^{n-1}$

7. (i) Let $f(x) = (x - a)\,(x - b)$ ... (i)

Differentiating (i) with respect to x, we get

$f'(x) = (x - a)\,(x - b)' + (x - a)'\,(x - b)$

$\Rightarrow\ f'(x) = (x - a) + (x - b) = 2x - a - b$

(ii) Let $f(x) = (ax^2 + b)^2$ ... (i)

Differentiating (i) with respect to x, we get

$f'(x) = 2(ax^2 + b) \times 2ax = 4ax\,(ax^2 + b)$

(iii) Let $f(x) = \frac{x-a}{x-b}$ ... (i)

Differentiating (i) with respect to x, we get

$$f'(x) = \frac{(x-b)\,(x-a)' - (x-a)\,(x-b)'}{(x-b)^2}$$

$$f'(x) = \frac{(x-b) - (x-a)}{(x-b)^2} = \frac{x - b - x + a}{(x-b)^2} = \frac{a-b}{(x-b)^2}$$

8. Let $f(x) = \frac{x^n - a^n}{x - a}$ ...(i), where a is a constant.

Differentiating (i) with respect to x, we get

$$f'(x) = \frac{(x-a)\,(x^n - a^n)' - (x^n - a^n)\cdot(x-a)'}{(x-a)^2}$$

$$= \frac{(x-a)\,(nx^{n-1} - 0) - (x^n - a^n)\cdot(1)}{(x-a)^2}$$

$$= \frac{nx^{n-1}(x-a) - x^n + a^n}{(x-a)^2} = \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

9. (i) Let $f(x) = 2x - \frac{3}{4}$ ... (1)

Differentiating (1) with respect to x, we get

$f'(x) = 2\cdot 1 - 0 = 2$

(ii) Let $f(x) = (5x^3 + 3x - 1)(x - 1)$ ... (1)

Differentiating (1) with respect to x, we get

$f'(x) = (5x^3 + 3x - 1)'(x - 1) + (5x^3 + 3x - 1)(x - 1)'$
$= (5\cdot 3x^2 + 3 - 0)(x - 1) + (5x^3 + 3x - 1)(1 - 0)$
$= (15x^2 + 3)(x - 1) + (5x^3 + 3x - 1)(1)$
$= 15x^3 + 3x - 15x^2 - 3 + 5x^3 + 3x - 1$
$= 20x^3 - 15x^2 + 6x - 4$

(iii) Let $f(x) = x^{-3}(5 + 3x)$ ... (1)

Differentiating (1) with respect to x, we get

$f'(x) = (x^{-3})'(5 + 3x) + (x^{-3})(5 + 3x)'$
$= (-3)x^{-3-1}(5 + 3x) + (x^{-3})(0 + 3)$
$= -3x^{-4}(5 + 3x) + x^{-3}\cdot(3)$
$= -15x^{-4} - 9x^{-3} + 3x^{-3}$
$= -15x^{-4} - 6x^{-3} = \frac{-15}{x^4} - \frac{6}{x^3}$

$\therefore \quad f'(x) = \frac{-3}{x^4}(5 + 2x)$

(iv) Let $f(x) = x^5(3 - 6x^{-9})$ ... (1)

Differentiating (1) with respect to x, we get

$f'(x) = (x^5)'(3 - 6x^{-9}) + x^5(3 - 6x^{-9})'$
$= 5x^4(3 - 6x^{-9}) + x^5(0 + 6\cdot 9x^{-10})$
$= 15x^4 - 30x^{-5} + 54x^{-5}$

$\therefore \quad f'(x) = 15x^4 + 24x^{-5} = 15x^4 + \frac{24}{x^5}$

(v) Let $f(x) = x^{-4}(3 - 4x^{-5})$ ... (1)

Differentiating (1) with respect to x, we get

$f'(x) = (x^{-4})'(3 - 4x^{-5}) + x^{-4}(3 - 4x^{-5})'$
$= -4x^{-5}(3 - 4x^{-5}) + x^{-4}(0 + 20x^{-6})$
$= -12x^{-5} + 16x^{-10} + 20x^{-10} = -12x^{-5} + 36x^{-10}$

$\therefore \quad f'(x) = \frac{-12}{x^5} + \frac{36}{x^{10}}$

(vi) Let $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$ ... (1)

Differentiating (1) with respect to x, we get

$$f'(x) = \left[\frac{(x+1)(2)' - (2)\cdot(x+1)'}{(x+1)^2}\right] - \left[\frac{(3x-1)(x^2)' - (x^2)(3x-1)'}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{(3x-1)(2x) - x^2(3)}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right] = \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x}{(3x-1)^2}\right]$$

$$\therefore \quad f'(x) = \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

10. Let $f(x) = \cos x$

We have $\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0}\left[\frac{-2\sin\left(\frac{x+h+x}{2}\right)\cdot\sin\left(\frac{x+h-x}{2}\right)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-2\sin\left(x+\frac{h}{2}\right)\cdot\sin\frac{h}{2}}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-\sin\left(x+\frac{h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]$$

$$= \lim_{h\to 0}\left[-\sin\left(x+\frac{h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]$$

$$= -\left\{\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\right\}\left\{\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right\} = -\sin x$$

$$\therefore \quad \frac{d}{dx}(\cos x) = -\sin x$$

11. (i) Let $f(x) = \sin x\cos x$ ... (1)

Differentiating (1) with respect to x, we get

$f'(x) = (\sin x)'\cos x + (\cos x)'\sin x$
$= \cos x\cdot\cos x + (-\sin x)\cdot\sin x$
$= \cos^2 x - \sin^2 x$

$\therefore \quad f'(x) = \cos 2x$

(ii) Let $f(x) = \sec x$

$\Rightarrow \quad f(x) = \frac{1}{\cos x} \Rightarrow f(x) = (\cos x)^{-1}$ ... (1)

Differentiating (1) with respect to x, we get

$f'(x) = -(\cos x)^{-2}\cdot(-\sin x)$

$\Rightarrow \quad f'(x) = \frac{\sin x}{\cos^2 x} \Rightarrow f'(x) = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$

$\therefore \quad f'(x) = \tan x\cdot\sec x.$

(iii) Let $f(x) = 5\sec x + 4\cos x$

$\Rightarrow \quad f(x) = \frac{5}{\cos x} + 4\cos x$

$\Rightarrow \quad f(x) = 5\cdot(\cos x)^{-1} + 4\cos x$ ... (1)

Differentiating (1) with respect to x, we get

$f'(x) = 5(-1)(\cos x)^{-2}(-\sin x) + 4(-\sin x)$

$$= \frac{5\sin x}{\cos^2 x} - 4\sin x = \frac{5\sin x}{\cos x}\cdot\frac{1}{\cos x} - 4\sin x$$

$\therefore \quad f'(x) = 5\tan x\cdot\sec x - 4\sin x$

(iv) Let $f(x) = \text{cosec } x$

$$\Rightarrow f(x) = \frac{1}{\sin x} \Rightarrow f(x) = (\sin x)^{-1} \quad \text{... (1)}$$

Differentiating (1) with respect to x, we get

$$f'(x) = (-1)(\sin x)^{-2} \cdot \cos x$$

$$= \frac{-\cos x}{\sin^2 x} = \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$\therefore \quad f'(x) = -\cot x \text{ cosec } x$

(v) Let $f(x) = 3 \cot x + 5 \text{ cosec } x$

$$\Rightarrow f(x) = \frac{3 \cos x}{\sin x} + \frac{5}{\sin x} \quad \text{... (1)}$$

Differentiating (1) with respect to x, we get

$$f'(x) = 3\left[\frac{\sin x (\cos x)' - \cos x \cdot (\sin x)'}{\sin^2 x}\right] + 5\left[\frac{\sin x (1)' - 1 \cdot (\sin x)'}{\sin^2 x}\right]$$

$$= 3\left[\frac{\sin x \cdot (-\sin x) - \cos x (\cos x)}{\sin^2 x}\right] + 5\left[\frac{0 - \cos x}{\sin^2 x}\right]$$

$$= 3\left[\frac{-\sin^2 x - \cos^2 x}{\sin^2 x}\right] + 5\left[\frac{-\cos x}{\sin^2 x}\right]$$

$$= -3\left[\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right] - 5\left[\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}\right]$$

$$= -3\left[\frac{1}{\sin^2 x}\right] - 5[\cot x \cdot \text{cosec } x]$$

$= -3 \text{ cosec}^2 x - 5 \cot x \cdot \text{cosec } x$

$= -\text{cosec } x [3 \text{ cosec } x + 5 \cot x]$

(vi) Let $f(x) = 5 \sin x - 6 \cos x + 7$ ... (1)

Differentiating (1) with respect to x, we get

$f'(x) = 5 \cos x - 6(-\sin x) + 0$

$\therefore \quad f'(x) = 5 \cos x + 6 \sin x$

(vii) Let $f(x) = 2 \tan x - 7 \sec x$ ... (1)

Differentiating (1) with respect to x, we get

$f'(x) = 2 \sec^2 x - 7 \sec x \tan x$

NCERT MISCELLANEOUS EXERCISE

1. (i) Let $f(x) = -x$

$$\text{We have, } \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h} = \lim_{h \to 0} \frac{-x - h + x}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

(ii) Let $f(x) = (-x)^{-1} \Rightarrow f(x) = \frac{-1}{x}$

$$\text{We have, } \frac{d}{dx}(f(x)) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h \to 0} \frac{\left[\frac{-1}{x+h} - \left(\frac{-1}{x}\right)\right]}{h} = \lim_{h \to 0} \frac{\left[\frac{-1}{x+h} + \frac{1}{x}\right]}{h}$$

$$= \lim_{h \to 0}\left[\frac{-x + x + h}{x(x+h)h}\right] = \lim_{h \to 0}\left[\frac{h}{hx(x+h)}\right] = \frac{1}{x^2}$$

(iii) Let $f(x) = \sin(x+1)$

$$\text{We have, } \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h+1) - \sin(x+1)}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{2\left[\cos\left(\frac{2(x+1)+h}{2}\right)\sin\frac{h}{2}\right]}{2 \times \left(\frac{h}{2}\right)}$$

$$= \lim_{h \to 0} \cos\left(x+1+\frac{h}{2}\right)\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)$$

$$= \left\{\lim_{h \to 0} \cos\left(x+1+\frac{h}{2}\right)\right\}\left\{\lim_{h \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}\right\}$$

$= \cos(x+1) \times (1) = \cos(x+1)$

(iv) Let $f(x) = \cos\left(x - \frac{\pi}{8}\right)$

$$\text{We have, } \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos\left(x+h-\frac{\pi}{8}\right) - \cos\left(x - \frac{\pi}{8}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{x+h-\frac{\pi}{8}+x-\frac{\pi}{8}}{2}\right)\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)}{h}$$

$$= \lim_{h \to 0}\left[\frac{-2\sin\left(x-\frac{\pi}{8}+\frac{h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{h}\right]$$

$$= \left\{-\lim_{h \to 0} \sin\left(x - \frac{\pi}{8} + \frac{h}{2}\right)\right\}\left\{\lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}$$

$$= -\sin\left(x - \frac{\pi}{8}\right)(1)$$

2. Let $f(x) = (x + a)$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = 1 + 0 = 1$$

3. Let $f(x) = (px + q)\left(\frac{r}{x} + s\right)$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = (px+q)'\left(\frac{r}{x}+s\right) + (px+q)\left(\frac{r}{x}+s\right)'$$

$$= (p+0)\left(\frac{r}{x}+s\right) + (px+q)\left(\frac{x(r)' - r(x)'}{x^2}+0\right)$$

$$= p\left(\frac{r}{x}+s\right) + (px+q)\left(\frac{0-r}{x^2}\right)$$

$$= p\left(\frac{r}{x}+s\right) - \frac{(px+q)r}{x^2}$$

$$= \frac{pr}{x} + ps - \frac{pr}{x} - \frac{qr}{x^2} = ps - \frac{qr}{x^2}$$

4. Let $f(x) = (ax+b)(cx+d)^2$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = (ax+b)[(cx+d)^2]' + (cx+d)^2(ax+b)'$$

$$= (ax+b)\cdot 2c(cx+d) + (cx+d)^2 \cdot a$$

$$= 2c(ax+b)(cx+d) + a(cx+d)^2$$

5. Let $f(x) = \frac{ax+b}{cx+d}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = \frac{(cx+d)(ax+b)' - (ax+b)(cx+d)'}{(cx+d)^2}$$

$$= \frac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2}$$

$$= \frac{acx+ad-acx-bc}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}$$

6. Let $f(x) = \frac{1+\frac{1}{x}}{1-\frac{1}{x}}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = \frac{\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x}\right)' - \left(1+\frac{1}{x}\right)\left(1-\frac{1}{x}\right)'}{\left(1-\frac{1}{x}\right)^2}$$

$$= \frac{\left(1-\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) - \left(1+\frac{1}{x}\right)\left(\frac{1}{x^2}\right)}{\left(1-\frac{1}{x}\right)^2}$$

$$= \frac{\frac{-1}{x^2}+\frac{1}{x^3}-\frac{1}{x^2}-\frac{1}{x^3}}{\left(1-\frac{1}{x}\right)^2} = \frac{\frac{-2}{x^2}}{\left(1-\frac{1}{x}\right)^2} = \frac{\frac{-2}{x^2}}{\frac{(x-1)^2}{x^2}}$$

$$= \frac{-2}{(x-1)^2},\ x \neq 0, 1$$

7. Let $f(x) = \frac{1}{ax^2+bx+c}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = \frac{(ax^2+bx+c)\cdot(1)' - 1\cdot(ax^2+bx+c)'}{(ax^2+bx+c)^2}$$

$$= \frac{(ax^2+bx+c)(0) - (2ax+b)}{(ax^2+bx+c)^2} = \frac{-2ax-b}{(ax^2+bx+c)^2}$$

8. Let $f(x) = \frac{ax+b}{px^2+qx+r}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = \frac{(px^2+qx+r)(ax+b)' - (ax+b)(px^2+qx+r)'}{(px^2+qx+r)^2}$$

$$= \frac{(px^2+qx+r)(a+0) - (ax+b)(2px+q+0)}{(px^2+qx+r)^2}$$

$$= \frac{px^2a + qxa + ra - 2pax^2 - axq - 2pxb - bq}{(px^2+qx+r)^2}$$

$$= \frac{-pax^2 + ra - 2pxb - bq}{(px^2+qx+r)^2}$$

9. Let $f(x) = \frac{px^2+qx+r}{ax+b}$

$\Rightarrow\ f(x) = (px^2+qx+r)(ax+b)^{-1}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = (2px+q)(ax+b)^{-1} + (px^2+qx+r)[(-1)(ax+b)^{-2}a]$$

$$= \frac{2px+q}{(ax+b)} - \frac{a(px^2+qx+r)}{(ax+b)^2}$$

$$= \frac{(2px+q)(ax+b) - apx^2 - aqx - ar}{(ax+b)^2}$$

$$= \frac{2px^2a + 2pxb + aqx + bq - apx^2 - aqx - ar}{(ax+b)^2}$$

$$= \frac{apx^2 + 2pxb + bq - ar}{(ax+b)^2}$$

10. Let $f(x) = \frac{a}{x^4} - \frac{b}{x^2} + \cos x$

$\Rightarrow\ f(x) = ax^{-4} - bx^{-2} + \cos x$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = a(-4)x^{-5} - b(-2)x^{-3} + (-\sin x)$$

$$= \frac{-4a}{x^5} + \frac{2b}{x^3} - \sin x$$

11. Let $f(x) = 4\sqrt{x} - 2 \ \Rightarrow\ f(x) = 4x^{1/2} - 2$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = \frac{1}{2}\times 4x^{\frac{1}{2}-1} - 0$$

$$= \frac{1}{2}\times 4x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}} = \frac{2}{x^{1/2}}$$

12. Let $f(x) = (ax + b)^n$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = n(ax+b)^{n-1}(a) = na(ax+b)^{n-1}$$

13. Let $f(x) = (ax + b)^n \cdot (cx + d)^m$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = [(ax+b)^n]'(cx+d)^m + (ax+b)^n \cdot [(cx+d)^m]'$$

$$= [n(ax+b)^{n-1}\cdot(a\cdot 1+0)]\cdot(cx+d)^m + (ax+b)^n\cdot[m(cx+d)^{m-1}\cdot(c\cdot 1+0)]$$

$$= [n(ax+b)^{n-1}\cdot a]\,[cx+d]^m + [ax+b]^n\,[m(cx+d)^{m-1}c]$$

$$\therefore \frac{d}{dx}\left\{(ax+b)^n(cx+d)^m\right\} = (ax+b)^{n-1}(cx+d)^{m-1}[na(cx+d) + mc(ax+b)]$$

14. Let $f(x) = \sin(x + a)$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = \cos(x+a)\cdot(1+0) = \cos(x+a)$$

15. Let $f(x) = \operatorname{cosec} x \cot x$

$$\Rightarrow f(x) = \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} \Rightarrow f(x) = \frac{\cos x}{\sin^2 x}$$

$$\Rightarrow f(x) = \cos x(\sin x)^{-2} \quad \text{... (i)}$$

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = (-\sin x)(\sin x)^{-2} + (-2)(\sin x)^{-3}\cdot\cos x\cos x$$

$$= -(\sin x)^{-1} - 2(\sin x)^{-3}\cdot\cos^2 x$$

$$= -(\sin x)^{-1} - \frac{2\cos^2 x}{\sin^3 x}$$

$$= -(\sin x)^{-1} - 2\cot^2 x \operatorname{cosec} x$$

$$= -\operatorname{cosec} x - 2\cot^2 x \operatorname{cosec} x$$

$$= -\operatorname{cosec} x\{1 + 2\cot^2 x\}$$

$$= -\operatorname{cosec} x\{1 + \cot^2 x + \cot^2 x\}$$

$$\therefore \frac{d}{dx}(\operatorname{cosec} x \cot x) = -\operatorname{cosec} x\{\operatorname{cosec}^2 x + \cot^2 x\}$$

$$= -\operatorname{cosec}^3 x - \operatorname{cosec} x \cot^2 x$$

16. Let $f(x) = \dfrac{\cos x}{1+\sin x}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = \frac{(1+\sin x)(\cos x)' - \cos x(1+\sin x)'}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)(-\sin x) - \cos x(0+\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$\therefore \frac{d}{dx}\left[\frac{\cos x}{1+\sin x}\right] = \frac{-1}{(1+\sin x)}$$

17. Let $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = \frac{(\sin x - \cos x)(\sin x + \cos x)' - (\sin x + \cos x)(\sin x - \cos x)'}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{\{\sin x\cos x - \sin^2 x - \cos^2 x + \cos x\sin x - [\sin x + \cos x]^2\}}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

18. Let $f(x) = \dfrac{\sec x - 1}{\sec x + 1}$

$$\Rightarrow f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} \Rightarrow f(x) = \frac{1-\cos x}{1+\cos x} \quad \text{...(i)}$$

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = \frac{(1+\cos x)(1-\cos x)' - (1-\cos x)(1+\cos x)'}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\sin x) - (1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\sin x + \sin x\cdot\cos x + \sin x - \sin x\cdot\cos x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2} = \frac{2\frac{1}{\operatorname{cosec} x}}{\left(1+\frac{1}{\sec x}\right)^2}$$

$$= \frac{\frac{2}{\operatorname{cosec} x}}{\frac{(\sec x+1)^2}{\sec^2 x}} = \frac{2}{\operatorname{cosec} x}\cdot\frac{\sec^2 x}{(\sec x+1)^2}$$

$$= \frac{2\sin x\cdot\frac{1}{\cos x}\cdot\sec x}{(\sec x+1)^2} = \frac{2\tan x\cdot\sec x}{(\sec x+1)^2}$$

19. Let $f(x) = \sin^n x$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = n\sin^{n-1} x\cos x$$

20. Let $f(x) = \dfrac{a + b\sin x}{c + d\cos x}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = \frac{(a+b\sin x)'(c+d\cos x) - (a+b\sin x)(c+d\cos x)'}{(c+d\cos x)^2}$$

$$= \frac{b\cos x\,(c + d\cos x) - (a + b\sin x)\,(-d\sin x)}{(c + d\cos x)^2}$$

$$= \frac{bc\cos x + bd\cos^2 x + ad\sin x + bd\sin^2 x}{(c + d\cos x)^2}$$

$$= \frac{bc\cos x + bd\,(\cos^2 x + \sin^2 x) + ad\sin x}{(c + d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd}{(c + d\cos x)^2}$$

21. Let $f(x) = \dfrac{\sin(x + a)}{\cos x}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}(f(x)) = \frac{\cos x\,(\sin(x+a))' - (\sin(x+a))(\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos x\cos(x + a) - \sin(x + a)\,(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos x\cos(x + a) + \sin(x + a)\sin x}{(\cos x)^2}$$

$$= \frac{\cos[x + a - x]}{(\cos x)^2} = \frac{\cos a}{(\cos x)^2}$$

22. Let $f(x) = x^4\,(5\sin x - 3\cos x)$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = (x^4)'\,[5\sin x - 3\cos x] + x^4[5\sin x - 3\cos x]'$$

$= 4x^3\,[5\sin x - 3\cos x] + x^4\,[5\cos x + 3\sin x]$

$= 20x^3\sin x - 12x^3\cos x + 5x^4\cos x + 3x^4\sin x$

$= x^3\sin x\,(20 + 3x) + x^3\cos x\,(5x - 12)$

23. Let $f(x) = (x^2 + 1)\cos x$

$\Rightarrow\ f(x) = x^2\cos x + \cos x$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = 2x\cos x + x^2\,(-\sin x) + (-\sin x)$$

$= 2x\cos x - x^2\sin x - \sin x$

24. Let $f(x) = (ax^2 + \sin x)\,(p + q\cos x)$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = (ax^2 + \sin x)'\,(p + q\cos x) + (p + q\cos x)'\,(ax^2 + \sin x)$$

$= (2ax + \cos x)\,(p + q\cos x) + (-q\sin x)(ax^2 + \sin x)$

25. Let $f(x) = (x + \cos x)\,(x - \tan x)$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = (x + \cos x)'\,(x - \tan x) + (x - \tan x)'(x + \cos x)$$

$= (1 - \sin x)\,(x - \tan x) + (1 - \sec^2 x)\,(x + \cos x)$

$= (1 - \sin x)\,(x - \tan x) + (-\tan^2 x)\,(x + \cos x)$

26. Let $f(x) = \dfrac{4x + 5\sin x}{3x + 7\cos x}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = \frac{(3x + 7\cos x)\,(4x + 5\sin x)' - (4x + 5\sin x)(3x + 7\cos x)'}{(3x + 7\cos x)^2}$$

$$= \frac{(3x + 7\cos x)(4 + 5\cos x) - (4x + 5\sin x)(3 - 7\sin x)}{(3x + 7\cos x)^2}$$

$$= \frac{12x + 15x\cos x + 28\cos x + 35\,[\cos^2 x + \sin^2 x] - 12x + 28x\sin x - 15\sin x}{(3x + 7\cos x)^2}$$

$$= \frac{35 + 15x\cos x + 28\cos x + 28x\sin x - 15\sin x}{(3x + 7\cos x)^2}$$

27. Let $f(x) = \dfrac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = \cos\frac{\pi}{4}\left\{\frac{\sin x\cdot(x^2)' - (x^2)\,(\sin x)'}{\sin^2 x}\right\}$$

$$= \cos\left(\frac{\pi}{4}\right)\left\{\frac{\sin x\cdot 2x - x^2\cos x}{\sin^2 x}\right\}$$

$$= x\cos\frac{\pi}{4}\left[\frac{2\sin x - x\cos x}{\sin^2 x}\right]$$

28. Let $f(x) = \dfrac{x}{1 + \tan x}$

$\Rightarrow\ f(x) = x\,(1 + \tan x)^{-1}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = 1\cdot(1 + \tan x)^{-1} + x(\sec^2 x)(-1)(1 + \tan x)^{-2}$$

$= (1 + \tan x)^{-1} - x\sec^2 x(1 + \tan x)^{-2}$

$$= \frac{1}{1 + \tan x} - \frac{x\sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{1 + \tan x - x\sec^2 x}{(1 + \tan x)^2}$$

29. Let $f(x) = (x + \sec x)\,(x - \tan x)$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = (x + \sec x)'\,(x - \tan x) + (x - \tan x)'\,(x + \sec x)$$

$= (1 + \sec x\tan x)\,(x - \tan x) + (1 - \sec^2 x)\,(x + \sec x)$

30. Let $f(x) = \dfrac{x}{\sin^n x}$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{d}{dx}[f(x)] = \frac{\sin^n x\cdot 1 - x\cdot n\sin^{n-1} x\cos x}{(\sin^n x)^2}$$

$$= \frac{(\sin x)^{n-1}(\sin x - nx\cos x)}{(\sin x)^{2n}}$$

$$= \frac{\sin x - nx\cos x}{(\sin x)^{2n-n+1}} = \frac{\sin x - nx\cos x}{(\sin x)^{n+1}}$$