Complex Numbers and Quadratic Equations

CHAPTER 5

EXAM DRILL

ANSWERS

1. **(d) :** Clearly, $i^{30} + i^{40} + i^{60} = i^{28+2} + i^{40} + i^{60}$

$= (i^4)^7 \cdot i^2 + (i^4)^{10} + (i^4)^{15} = 1 \cdot i^2 + 1^{10} + 1^{15}$ $[\because i^4 = 1]$

$= -1 + 1 + 1$

$= 1$

2. **(a) :** Clearly,

$$\left[i^{41} + \frac{1}{i^{257}}\right]^2 = \left[(i^2)^{20} \cdot i + \frac{1}{(i^2)^{128} \cdot i}\right]^2 = \left[i + \frac{1}{i}\right]^2$$

$= [i - i]^2 = 0^2 = 0$

3. **(b) :** Let $z = 2 - 3i$, then

$\bar{z} = 2 + 3i$ and $|z|^2 = (2)^2 + (-3)^2 = 13$

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

4. **(c) :** Multiplicative inverse of $\dfrac{3+4i}{4-5i} = \dfrac{4-5i}{3+4i}$

$$= \frac{4-5i}{3+4i} \times \frac{3-4i}{3-4i} = \frac{(12-20)+i(-15-16)}{9+16}$$

$$= \frac{-8}{25} - i\frac{31}{25}$$

5. Modulus of $1 + i = \sqrt{1^2 + 1^2} = \sqrt{2}$

6. Required coordinates = (2, 3)

7. Conjugate of $2i$ is $-2i$.

8. Let $\bar{z} = \dfrac{1}{i-1}$

We know $z = \overline{(\bar{z})}$, therefore $z = \dfrac{1}{\bar{i}-1} = \dfrac{1}{-i-1} = \dfrac{-1}{i+1}$

9 **(i)** $x + y = 6$ ...(i) and $x - y = 2$ ...(ii)

Solving (i) and (ii) equation, we get

$x = 4$ and $y = 2$

So, $xy = 4(2) = 8$

(ii) $(1 + i)\, z = (1 - i)\, \bar{z}$

$\Rightarrow$ $(1 + i)(1 - i)\, z = (1 - i)^2\, z$

$\Rightarrow$ $(1 - i^2)\, z = (1 - 2i + i^2)\, \bar{z}$

$\Rightarrow$ $2z = -2i\bar{z}$ $[\because i^2 = -1]$

$\Rightarrow$ $z = -i\bar{z}$ ...(i)

$z = x + iy \therefore \bar{z} = x - iy$

By (i), we get

$x + iy = -i(x - iy)$

$= -ix - y$

Equating coefficients, $y = -x$

$\Rightarrow$ $z = x - ix$

$= x(1 - i), x \in R$

10. Consider, $4x + (3x - y)i = 3 - 6i$

Equating real and imaginary parts, we get

$4x = 3$ and $3x - y = -6$

$$\Rightarrow \quad x = \frac{3}{4} \text{ and } \frac{9}{4} - y = -6 \Rightarrow y = \frac{33}{4}$$

11. We have, $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i} = \dfrac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \dfrac{1+\sqrt{2}i}{1+\sqrt{2}i}$

$$= \frac{5+5\sqrt{2}i+\sqrt{2}i+2i^2}{(1)^2-(\sqrt{2}i)^2}$$

$$= \frac{5+6\sqrt{2}i-2}{1-2i^2} = \frac{3+6\sqrt{2}i}{1+2} = \frac{3+6\sqrt{2}i}{3} = 1+2\sqrt{2}i$$

OR

Let $z_1 = -\sqrt{3} + \sqrt{-2} = -\sqrt{3} + i\sqrt{2}$

and $z_2 = 2\sqrt{3} - i$

Then $z_1 + z_2 = (-\sqrt{3} + i\sqrt{2}) + (2\sqrt{3} - i)$

$= (-\sqrt{3} + 2\sqrt{3}) + i(\sqrt{2} - 1) = \sqrt{3} + i(\sqrt{2} - 1)$

and $z_1 z_2 = (-\sqrt{3} + i\sqrt{2})(2\sqrt{3} - i)$

$= (-6 + \sqrt{2}) + i(\sqrt{3} + 2\sqrt{6})$

12. Consider, $i = 0 + i = \dfrac{1}{2}(0 + 2i)$

$$= \frac{1}{2}(1^2 + i^2 + 2i) = \frac{1}{2}(1+i)^2$$

$$\therefore \quad \sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i)$$

13. We have, $\dfrac{1+i\cos\theta}{1-2i\cos\theta} = \dfrac{1+i\cos\theta}{1-2i\cos\theta} \times \dfrac{1+2i\cos\theta}{1+2i\cos\theta}$

$$= \frac{1+2i\cos\theta+i\cos\theta+2i^2\cos^2\theta}{1-4i^2\cos^2\theta}$$

$$= \frac{1-2\cos^2\theta+3i\cos\theta}{1+4\cos^2\theta} = \frac{1-2\cos^2\theta}{1+4\cos^2\theta} + \frac{3\cos\theta}{1+4\cos^2\theta}i$$

For purely real, imaginary part must be zero.

$$\therefore \quad \frac{3\cos\theta}{1+4\cos^2\theta} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

14. We have,

$$\left(\frac{1}{3}+3i\right)^3 = \left(\frac{1}{3}\right)^3 + 3\left(\frac{1}{3}\right)^2(3i) + 3\left(\frac{1}{3}\right)(3i)^2 + (3i)^3$$

$= \frac{1}{27} + i + 9(-1) + 27i^3$

$= \frac{1}{27} + i - 9 + 27i(-1) = \frac{1}{27} + i - 9 - 27i$

$= \left(\frac{1}{27} - 9\right) + (1-27)i = -\frac{242}{27} - 26i$

$= a + ib$, where $a = \frac{-242}{27}$ and $b = -26$

15. We have, $\frac{(1+i)^2}{2-i} = x + iy$

$\Rightarrow \frac{(1+i^2+2i)}{2-i} = x+iy \Rightarrow \frac{2i}{2-i} = x+iy$

$\Rightarrow \frac{2i(2+i)}{(2-i)(2+i)} = x+iy \Rightarrow \frac{4i+2i^2}{4-i^2} = x+iy$

$\Rightarrow \frac{4i-2}{4+1} = x+iy \Rightarrow \frac{-2}{5} + \frac{4i}{5} = x+iy$

Equating real and imaginary parts, we get
$x = -2/5$ and $y = 4/5$

$\Rightarrow x + y = \frac{-2}{5} + \frac{4}{5} = \frac{2}{5}$

16. Let $z = \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{1+1} = \frac{1}{2} - \frac{i}{2}$

$\therefore$ Modulus of $z = |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$

Also, conjugate of $z = \bar{z} = \frac{1}{2} + \frac{i}{2}$

17. Let $z = \frac{2+i}{4i+(1+i)^2} = \frac{2+i}{4i+1+i^2+2i} = \frac{2+i}{6i}$

$(\because i^2 = -1)$

$= \frac{2+i}{6i} \times \frac{i}{i} = \frac{2i+i^2}{6i^2} = \frac{2i}{-6} + \frac{i^2}{6i^2} = \frac{1}{6} - \frac{1}{3}i$

Now, conjugate of $z = \bar{z} = \frac{1}{6} + \frac{1}{3}i$

18. We have, $p + iq = \frac{(a+i)^2}{2a-i}$ ...(i)

Taking conjugate on both sides, we get

$p - iq = \frac{(a-i)^2}{2a+i}$ ...(ii)

Multiplying (i) and (ii), we get

$(p+iq)(p-iq) = \frac{(a+i)^2}{(2a-i)} \times \frac{(a-i)^2}{(2a+i)}$

$\Rightarrow p^2 - (iq)^2 = \frac{[(a+i)(a-i)]^2}{(2a-i)(2a+i)}$

$\Rightarrow p^2 + q^2 = \frac{(a^2+1)^2}{4a^2+1}$

19. Let $z = x + iy$, $\bar{z} = x - iy$

$\therefore \frac{\bar{z}+2}{\bar{z}-1} = \frac{x-iy+2}{x-iy-1}$

$= \frac{[(x+2)-iy][(x-1)+iy]}{[(x-1)-iy][(x-1)+iy]}$

$= \frac{(x-1)(x+2) - iy(x-1) + iy(x+2) + y^2}{(x-1)^2 + y^2}$

$= \frac{(x-1)(x+2) + y^2 + i[(x+2)y - (x-1)y]}{(x-1)^2 + y^2}$

Since $\text{Re}\left(\frac{\bar{z}+2}{\bar{z}-1}\right) = 4$

$\Rightarrow \frac{(x-1)(x+2) + y^2}{(x-1)^2 + y^2} = 4$

$\Rightarrow x^2 - x + 2x - 2 + y^2 = 4(x^2 - 2x + 1 + y^2)$

$\Rightarrow 3x^2 + 3y^2 - 9x + 6 = 0$,

which is an equation of a circle.

20. We have, $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$

$= \left\{\frac{1+i-2+8i}{(1-4i)(1+i)}\right\}\left\{\frac{(3-4i)(5-i)}{(5+i)(5-i)}\right\}$

$= \left\{\frac{-1+9i}{1+i-4i-4i^2}\right\}\left\{\frac{15-3i-20i+4i^2}{25-i^2}\right\}$

$= \left\{\frac{-1+9i}{5-3i}\right\}\left\{\frac{11-23i}{26}\right\}$

$= \left\{\frac{(-1+9i)(5+3i)}{25-9i^2}\right\}\left\{\frac{11-23i}{26}\right\}$

$= \left\{\frac{-5-3i+45i+27i^2}{25+9}\right\}\left\{\frac{11-23i}{26}\right\}$

$= \left\{\frac{-32+42i}{34}\right\}\left\{\frac{11-23i}{26}\right\} = \left\{\frac{-16+21i}{17}\right\}\left\{\frac{11-23i}{26}\right\}$

$= \frac{-176+368i+231i-483i^2}{442}$

$= \frac{307+599i}{442} = \frac{307}{442} + \frac{599}{442}i$

$\therefore$ Magnitude $= \sqrt{\left(\frac{307}{442}\right)^2 + \left(\frac{599}{442}\right)^2}$

$= \sqrt{\frac{94249+358801}{(442)^2}} = \sqrt{\frac{453050}{195364}} = \sqrt{\frac{1025}{442}}$

and conjugate $= \frac{307}{442} - \frac{599}{442}i$