Complex Numbers and Quadratic Equations

CHAPTER 5

NCERT FOCUS

ANSWERS

EXERCISE - 5.1

1. $(5i)\left(-\frac{3}{5}i\right) = -3i^2 = -3(-1)$ $[\because i^2 = -1]$
$= 3 = 3 + 0i$

2. $i^9 + i^{19} = (i^2)^4\cdot i + (i^2)^9\cdot i = (-1)^4\cdot i + (-1)^9\cdot i$
$= i + (-1)i = i - i = 0 = 0 + 0i$

3. $i^{-39} = \frac{1}{i^{39}} = \frac{1}{(i^2)^{19}\cdot i}$

$= \frac{1}{(-1)^{19}\cdot i}$ $[\because i^2 = -1]$

$= \frac{1}{(-1)\cdot i} = \frac{1}{-i} = -\frac{1}{i}\times\frac{i}{i} = \frac{-i}{i^2}$

$= \frac{-i}{-1} = i = 0 + 1i$

4. $3(7 + i7) + i(7 + i7) = 21 + 21i + 7i + 7i^2$
$= 21 + (21 + 7)i + (-1)7 = 21 - 7 + 28i$
$= 14 + 28i$

5. $(1 - i) - (-1 + i6) = 1 - i + 1 - 6i$
$= (1 + 1) - i(1 + 6) = 2 - 7i$

6. $\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right) = \frac{1}{5} + i\frac{2}{5} - 4 - i\frac{5}{2}$

$= \left(\frac{1}{5} - 4\right) + i\left(\frac{2}{5} - \frac{5}{2}\right)$

$= \left(\frac{1-20}{5}\right) + i\left(\frac{4-25}{10}\right) = \frac{-19}{5} - \frac{21}{10}i$

$= \left(\frac{-19}{5}\right) + \left(\frac{-21}{10}\right)i$

7. $\left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right)\right] - \left(-\frac{4}{3} + i\right)$

$= \left[\left(\frac{1}{3} + 4\right) + i\left(\frac{7}{3} + \frac{1}{3}\right)\right] - \left(\frac{-4}{3} + i\right)$

$= \left(\frac{1+12}{3}\right) + i\left(\frac{7+1}{3}\right) + \frac{4}{3} - i$

$= \frac{13}{3} + i\frac{8}{3} + \frac{4}{3} - i$

$= \frac{13}{3} + \frac{4}{3} + i\left(\frac{8}{3} - 1\right) = \frac{13+4}{3} + i\left(\frac{8-3}{3}\right) = \frac{17}{3} + \frac{5}{3}i$

8. $(1 - i)^4 = [(1 - i)^2]^2 = [1 - 2i + i^2]^2$
$= [1 - 2i + (-1)]^2$
$= (-2i)^2 = 4i^2 = 4(-1) = -4 = -4 + 0i$

9. $\left(\frac{1}{3} + 3i\right)^3 = \left(\frac{1}{3}\right)^3 + (3i)^3 + 3\left(\frac{1}{3}\right)^2(3i) + 3\left(\frac{1}{3}\right)(3i)^2$

$= \frac{1}{27} + 27i^3 + i + 9i^2 = \frac{1}{27} + 27(-i) + i + 9(-1)$

$= \frac{1}{27} - 9 + i(1 - 27) = \frac{1-243}{27} + i(-26)$

$= \left(\frac{-242}{27}\right) + (-26)i$

10. $\left(-2 - \frac{1}{3}i\right)^3 = (-2)^3 + \left(-\frac{1}{3}i\right)^3 + 3(-2)^2\left(\frac{-1}{3}i\right)$
$+ 3(-2)\left(\frac{-1}{3}i\right)^2$

$= -8 - \frac{1}{27}i^3 - 4i - \frac{2}{3}i^2 = -8 - \frac{1}{27}(-i) - 4i - \frac{2}{3}(-1)$

$= -8 + \frac{1}{27}i - 4i + \frac{2}{3} = \left(-8 + \frac{2}{3}\right) + \left(\frac{1}{27} - 4\right)i$

$= \left(\frac{-24+2}{3}\right) + \left(\frac{1-108}{27}\right)i = \left(\frac{-22}{3}\right) + \left(\frac{-107}{27}\right)i$

11. Let $z = 4 - 3i$. Then $\bar{z} = 4 + 3i$ ...(i)

and $|z|^2 = (4)^2 + (-3)^2 = 25$ ...(ii)

$\therefore$ The multiplicative inverse of $4 - 3i$ is given by

$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4+3i}{25}$ [By using (i) and (ii)]

$= \frac{4}{25} + i\frac{3}{25}$

12. Let $z = \sqrt{5} + 3i$. Then $\bar{z} = \sqrt{5} - 3i$ ...(i)

and $|z|^2 = (\sqrt{5})^2 + (3)^2 = 14$ ...(ii)

$\therefore$ The multiplicative inverse of $\sqrt{5} + 3i$ is given by

$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5} - 3i}{14}$ [By using (i) and (ii)]

$= \frac{\sqrt{5}}{14} + i\left(\frac{-3}{14}\right)$

13. Let $z = -i = 0 - i$. Then $\bar{z} = 0 + i$ ...(i)

and $|z|^2 = (0)^2 + (-1)^2 = 1$ ...(ii)

$\therefore$ The multiplicative inverse of $-i$ is given by

$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{0+i}{1} = i$ [By using (i) and (ii)]

14. We have, $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$

$= \frac{(3)^2-(i\sqrt{5})^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i}$

$= \frac{9-5i^2}{2\sqrt{2}i} = \frac{9-5(-1)}{2\sqrt{2}\,i} = \frac{9+5}{2\sqrt{2}i}$

$= \frac{14}{2\sqrt{2}i} = \frac{7}{\sqrt{2}i} \times \frac{i}{i} = \frac{7i}{\sqrt{2}\,i^2}$

$= \frac{7i}{\sqrt{2}(-1)} = \frac{-7i}{\sqrt{2}} = \frac{-7\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}i$

$= \frac{-7\sqrt{2}i}{2} = 0 + i\left(\frac{-7\sqrt{2}}{2}\right)$

NCERT MISCELLANEOUS EXERCISE

1. We have, $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3 = \left[(i^2)^9 + \frac{1}{(i^2)^{12}}\cdot\frac{1}{i}\right]^3$

$= \left[(-1)^9 + \frac{1}{(-1)^{12}}\cdot\frac{1}{i}\right]^3 = \left[-1+\frac{1}{i}\right]^3$

$= (-1)^3 + \left(\frac{1}{i}\right)^3 + 3(-1)^2\left(\frac{1}{i}\right) + 3(-1)\left(\frac{1}{i}\right)^2$

$= -1 + \frac{1}{-i} + \frac{3}{i} - \frac{3}{i^2} = -1 + \frac{-1}{i} + \frac{3}{i} - \frac{3}{-1}$

$= -1 + 3 + \frac{1}{i}(3-1) = 2 + \frac{2}{i}\times\frac{i}{i}$

$= 2 - 2i$

2. Let $z_1 = a + ib$ and $z_2 = c + id$, where $a, b, c, d \in R$

Then, $z_1z_2 = (a+ib)(c+id) = (ac-bd) + i(bc+ad)$

$\text{Re}(z_1z_2) = ac - bd = \text{Re } z_1 \text{ Re } z_2 - \text{Im } z_1 \text{ Im } z_2$

Hence proved.

3. We have,

$\left(\frac{1}{1-4i}-\frac{2}{1+i}\right) = \frac{1+i-2(1-4i)}{(1-4i)(1+i)}$

$= \frac{1+i-2+8i}{1+i-4i+4} = \frac{-1+9i}{5-3i}$

$\Rightarrow \left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \left(\frac{-1+9i}{5-3i}\right)\left(\frac{3-4i}{5+i}\right)$

$= \frac{-3+4i+27i+36}{25+5i-15i+3} = \frac{33+31i}{28-10i}\times\frac{28+10i}{28+10i}$

$= \frac{924+330i+868i-310}{784+100}$

$= \frac{614+1198i}{884} = \frac{614}{884}+\frac{1198}{884}i = \frac{307}{442}+\frac{599}{442}i$

4. We have $x - iy = \sqrt{\frac{a-ib}{c-id}}$

Squaring both sides, we get

$(x-iy)^2 = \frac{a-ib}{c-id} \Rightarrow |(x-iy)^2| = \left|\frac{a-ib}{c-id}\right|$

$\Rightarrow |x-iy|^2 = \frac{|a-ib|}{|c-id|}$

$\Rightarrow \left(\sqrt{x^2+y^2}\right)^2 = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$

$\Rightarrow x^2+y^2 = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$

Squaring both sides, we get

$(x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$

5. We have, $z_1 = 2 - i$ and $z_2 = 1 + i$

$z_1 + z_2 + 1 = (2-i) + (1+i) + 1 = 4 + 0i$

and $z_1 - z_2 + 1 = (2-i) - (1+i) + 1 = 2 - 2i$

$\therefore \left|\frac{z_1+z_2+1}{z_1-z_2+1}\right| = \frac{|z_1+z_2+1|}{|z_1-z_2+1|}$

$= \frac{|4+0i|}{|2-2i|} = \frac{\sqrt{4^2+0^2}}{\sqrt{4+4}} = \frac{4}{2\sqrt{2}} = \sqrt{2}$

6. We have, $a+ib = \frac{(x+i)^2}{2x^2+1}$

$\therefore |a+ib| = \left|\frac{(x+i)^2}{2x^2+1}\right| = \frac{|(x+i)^2|}{|2x^2+1|} = \frac{|x+i|^2}{|2x^2+1|}$

$\Rightarrow \sqrt{a^2+b^2} = \frac{\left(\sqrt{x^2+1^2}\right)^2}{\sqrt{(2x^2+1)^2}}$

Squaring both sides, we get

$a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$

7. We have, $z_1 = 2 - i$ and $z_2 = -2 + i$

(i) $z_1z_2 = (2-i)(-2+i) = -4 + 2i + 2i - i^2$

$= -4 + 1 + 4i = -3 + 4i$

Also, $\bar{z}_1 = 2 + i$

$\therefore \frac{z_1z_2}{\bar{z}_1} = \frac{-3+4i}{2+i} = \frac{-3+4i}{2+i}\times\frac{2-i}{2-i}$

$$= \frac{-6+3i+8i+4}{4+1} = \frac{-2+11i}{5} = \frac{-2}{5} + \frac{11}{5}i$$

$$\therefore \quad \text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = -\frac{2}{5}$$

(ii) $z_1 \bar{z}_1 = (2-i)(2+i) = 4+1 = 5$

Now, $\frac{1}{z_1\bar{z}_1} = \frac{1}{5} = \frac{1}{5} + 0i \Rightarrow \text{Im}\left(\frac{1}{z_1\bar{z}_1}\right) = 0$

8. We have, $(x-iy)(3+5i) = \overline{-6-24i}$

$\Rightarrow \quad 3x + 5xi - 3yi + 5y = -6 + 24i$

$\Rightarrow \quad (3x+5y) + (5x-3y)i = -6 + 24i$

Comparing real and imaginary parts, we get

$3x + 5y = -6,\ 5x - 3y = 24$

Solving above equations, we get

$x = 3$ and $y = -3$

9. We have, $\frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}$

$$= \frac{(1-1+2i)-(1-1-2i)}{1+1}$$

$$= \frac{2i+2i}{2} = \frac{4i}{2} = 0 + 2i$$

$$\therefore \quad \left|\frac{1+i}{1-i} - \frac{1-i}{1+i}\right| = \sqrt{2^2} = 2$$

10. We have, $(x+iy)^3 = u + iv$

$(x+iy)^3 = x^3 + (iy)^3 + 3x^2(iy) + 3x(iy)^2$

$= x^3 - y^3 i + 3x^2yi - 3xy^2$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

$\Rightarrow \quad (x^3 - 3xy^2) + i(3x^2y - y^3) = u + iv$

Comparing the real and imaginary parts, we get

$u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

$\Rightarrow \quad u = x(x^2 - 3y^2)$ and $v = y(3x^2 - y^2)$

$\Rightarrow \quad \frac{u}{x} = x^2 - 3y^2$ and $\frac{v}{y} = 3x^2 - y^2$

$\therefore \quad \frac{u}{x} + \frac{v}{y} = 4x^2 - 4y^2 = 4(x^2 - y^2)$

11. We have, $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = \frac{|\beta-\alpha|^2}{|1-\bar{\alpha}\beta|^2}$

$$= \frac{(\beta-\alpha)\overline{(\beta-\alpha)}}{(1-\bar{\alpha}\beta)\overline{(1-\bar{\alpha}\beta)}}$$

$$= \frac{(\beta-\alpha)(\bar{\beta}-\bar{\alpha})}{(1-\bar{\alpha}\beta)(1-\alpha\bar{\beta})}$$

$$= \frac{\beta\bar{\beta} - \beta\bar{\alpha} - \alpha\bar{\beta} + \alpha\bar{\alpha}}{1 - \alpha\bar{\beta} - \bar{\alpha}\beta + \alpha\bar{\alpha}\,\beta\bar{\beta}}$$

$$= \frac{1 - \bar{\alpha}\beta - \alpha\bar{\beta} + \alpha\bar{\alpha}}{1 - \bar{\alpha}\beta - \alpha\bar{\beta} + \bar{\alpha}\alpha} = 1 \quad [\because |\beta| = 1 \Rightarrow |\beta|^2 = 1 \Rightarrow \beta\bar{\beta} = 1]$$

$$\therefore \quad \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = 1 \Rightarrow \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1.$$

12. We have, $|(1-i)|^x = 2^x$

$\Rightarrow \quad (\sqrt{1+1})^x = 2^x \Rightarrow (\sqrt{2})^x = 2^x$

$\Rightarrow \quad (2^{1/2})^x = 2^x \Rightarrow 2^{x/2} = 2^x$

$\Rightarrow \quad \frac{x}{2} = x \Rightarrow 2x = x \Rightarrow x = 0$

$\therefore$ The number of non-zero integral solution is 0.

13. We have, $(a+ib)(c+id)(e+if)(g+ih) = A + iB$

$\therefore \quad |(a+ib)(c+id)(e+if)(g+ih)| = |A+iB|$

$\Rightarrow \quad |a+ib|\,|c+id|\,|e+if|\,|g+ih| = |A+iB|$

$\Rightarrow \quad \sqrt{a^2+b^2}\sqrt{c^2+d^2}\sqrt{e^2+f^2}\sqrt{g^2+h^2} = \sqrt{A^2+B^2}$

Squaring both sides, we get

$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2 + B^2$

14. We have, $\left(\frac{1+i}{1-i}\right)^m = -1 \Rightarrow \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^m = -1$

$$\Rightarrow \quad \left(\frac{1-1+2i}{1+1}\right)^m = -1$$

$$\Rightarrow \quad \left(\frac{2i}{2}\right)^m = -1 \Rightarrow (i)^m = -1$$

$\Rightarrow \quad m = 4K + 2$

$\therefore$ The least positive integral value of m is 2.