Binomial Theorem

EXAM DRILL

SOLUTIONS

1. **(a) :** Since $n = 8$, which is even.

$\therefore$ Required number of terms $= \left(\frac{8}{2}\right) = 4$.

2. **(b) :** We know, $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$

$\therefore$ $C_1 + C_2 + \ldots + C_n = 2^n - C_0 = 2^n - 1$.

3. **(d) :** $(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + \ldots + a_{12}x^{12}$

On putting $x = 1$ and $x = -1$ in the given expansion and adding the results, we get

$64 = 2(1 + a_2 + a_4 + \ldots + a_{12})$

$\Rightarrow$ $a_2 + a_4 + a_6 + \ldots + a_{12} = 31$.

4. **(b) :** Coefficient of $(r + 1)^{\text{th}}$ term in the expansion of $(1 + x)^n$ is ${}^nC_r$.

$\therefore$ Coefficient of 8^{th} term in the expansion of $(1 + x)^{10}$ is

${}^{10}C_7 = {}^{10}C_3 = \frac{10\times9\times8}{3\times2\times1} = 120$.

5. The total number of terms in the expansion of $(x + a)^{51} - (x - a)^{51}$ after simplification is 26.

$\left[\because \text{ Number of terms} = \left(\frac{51+1}{2}\right) = \frac{52}{2} = 26\right]$

6. Consider $25^{15} = (26 - 1)^{15}$

$= {}^{15}C_0 26^{15} - {}^{15}C_1 26^{14} + \ldots\ldots - {}^{15}C_{15}$

$= {}^{15}C_0 26^{15} - {}^{15}C_1 26^{14} + \ldots\ldots -1 - 13 + 13$

$= {}^{15}C_0 26^{15} - {}^{15}C_1 26^{14} + \ldots\ldots - 13 + 12$

$= 13k + 12$, where $k \in N$

Hence, when 25^{15} is divided by 13, then remainder will be 12.

7. Clearly, $(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_nb^n$

Now, coefficient of fourth term $= {}^nC_3$

and coefficient of thirteenth term $= {}^nC_{12}$

Now, according to given condition, we have

${}^nC_3 = {}^nC_{12}$

$\Rightarrow$ $n = 12 + 3 = 15$ $\quad [\because {}^nC_x = {}^nC_y \Rightarrow x = y \text{ or } x + y = n]$

8. We know, $C_0 + C_1 + \ldots + C_n = 2^n$

$\therefore$ $C_0 + C_1 + \ldots + C_9 = 2^9 = 512$

9. $\left(\frac{2x}{3} - \frac{3}{2x}\right)^4 = {}^4C_0\left(\frac{2x}{3}\right)^4 - {}^4C_1\left(\frac{2x}{3}\right)^3\left(\frac{3}{2x}\right) + {}^4C_2\left(\frac{2x}{3}\right)^2\left(\frac{3}{2x}\right)^2 - {}^4C_3\left(\frac{2x}{3}\right)\left(\frac{3}{2x}\right)^3 + {}^4C_4\left(\frac{3}{2x}\right)^4$

$= \frac{16x^4}{81} - 4\cdot\frac{2^3x^3}{3^3}\cdot\frac{3}{2x} + 6\cdot\frac{2^2\cdot x^2}{3^2}\cdot\frac{3^2}{2^2\cdot x^2} - 4\cdot\frac{2x}{3}\cdot\frac{3^3}{2^3x^3} + \frac{3^4}{2^4x^4}$

$= \frac{16}{81}x^4 - \frac{16x^2}{9} + 6 - \frac{9}{x^2} + \frac{81}{16x^4}$.

10. We have, $\left(3x - \frac{y^3}{6}\right)^4 = {}^4C_0(3x)^4 + {}^4C_1(3x)^3\left(\frac{-y^3}{6}\right)^1 + {}^4C_2(3x)^2\left(\frac{-y^3}{6}\right)^2 + {}^4C_3(3x)\left(\frac{-y^3}{6}\right)^3 + {}^4C_4\left(\frac{-y^3}{6}\right)^4$

$\therefore$ $T_4 = {}^4C_3\,(3x)\left(-\frac{y^3}{6}\right)^3$

$= -4 \times 3x \times \frac{y^9}{216} = \frac{-xy^9}{18}$

Thus 4^{th} term in the expansion of $\left(3x - \frac{y^3}{6}\right)^4$ is $\frac{-xy^9}{18}$.

11. We know that,

$(x + a)^n = {}^nC_0\,x^n + {}^nC_1\,x^{n-1}a + {}^nC_2\,x^{n-2}a^2 \ldots + {}^nC_n\,a^n$

(i) **(a) :** Now, $T_2 = 10$

$\Rightarrow$ ${}^nC_1\,x^{n-1}a = 10$ ...(i)

Similarly, ${}^nC_2\,x^{n-2}a^2 = 40$ ...(ii)

${}^nC_3\,x^{n-3}a^3 = 80$ ...(iii)

When third term is divided by the second term, we get

$\frac{T_3}{T_2} = \frac{{}^nC_2\cdot x^{n-2}\cdot a^2}{{}^nC_1\cdot x^{n-1}\cdot a} = \frac{40}{10} = \frac{(n-1)!\,a}{(n-2)!\,x} = 4\times2!$

$\Rightarrow$ $\frac{a}{x} = \frac{8}{n-1}$

(ii) **(c) :** On dividing (iii) by (ii), we get

$\frac{(n-2)!\,a}{(n-3)!\,x} = 2\times3$

$\Rightarrow$ $\frac{a}{x} = \frac{6}{n-2}$

On comparing, $\frac{8}{n-1} = \frac{6}{n-2}$

$\Rightarrow$ $8(n - 2) = 6(n - 1)$

$\Rightarrow$ $8n - 16 = 6n - 6$

$\Rightarrow$ $2n = 10$

$\Rightarrow$ $n = 5$.

(iii) **(b) :** As $n = 5$, $\frac{a}{x} = 2$

By (i), we have $5 \cdot x^4 a = 10$

$x^4 a = 2$

$\Rightarrow \quad x = 1$ and $a = 2 \qquad \left(\because \frac{a}{x} = 2\right)$

(iv) (a) : Since, $n = 5$, $x = 1$ and $a = 2$

$\therefore \quad (1 + 2)^5 = 1 + 5 \times 2 + 10 \times 2^2 + 10 \times 2^3 + 5 \times 2^4 + 2^5$

$\therefore$ Coefficient of a^2 *i.e.*, $2^2 = 10$.

(v) (d) : $(1 + 2)^5 = 1 + 5 \times 2 + 10 \times 2^2 + 10 \times 2^3 + 5 \times 16 + 32$

$= 1 + 10 + 40 + 80 + 80 + 32$

$= 243$.

12. (i) $(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

$(1 - x)^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$

Now, $(1 + x)^5 + (1 - x)^5 = 2[1 + 10x^2 + 5x^4]$.

(ii) Now, $(1 + ax)^4 = 1 + 4 \cdot ax + 6(ax)^2 + 4(ax)^3 + a^4x^4$

Given that the coefficient of x^2 and x^3 are equal

$\Rightarrow \quad 6a^2 = 4a^3$

$\Rightarrow \quad 4a = 6$

$\Rightarrow \quad a = \frac{6}{4} = \frac{3}{2}$.

13. Consider $(9.9)^3 = (10 - 0.1)^3$

$= {}^3C_0(10)^3 - {}^3C_1(10)^2(0.1) + {}^3C_2(10)(0.1)^2 - {}^3C_3(0.1)^3$

$= 1000 - 3 \cdot 100 \times \frac{1}{10} + 3 \cdot 10 \cdot \frac{1}{100} - \frac{1}{1000}$

$= 1000 - 30 + 0.3 - 0.001$

$= 1000.3 - 30.001 = 970.299$.

14. Putting $\sqrt{1-x^2} = y$, in the given expression, we get

$(x^2 - y)^4 + (x^2 + y)^4 = 2[x^8 + {}^4C_2 x^4 y^2 + {}^4C_4 y^4]$

$= 2\left[x^8 + \frac{4\times3}{2\times1}x^4 \cdot (1-x^2) + (1-x^2)^2\right]$

$= 2[x^8 + 6x^4(1 - x^2) + (1 - 2x^2 + x^4)]$

$= 2x^8 - 12x^6 + 14x^4 - 4x^2 + 2$.

15. Let the binomial expansion be

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$

According to question,

$nx = 10 \Rightarrow n^2x^2 = 100 \qquad \ldots$(i)

and $\frac{n(n-1)}{2!}x^2 = 40 \qquad \ldots$(ii)

Dividing (ii) by (i), we get

$\frac{n(n-1)}{2!}x^2 \cdot \frac{1}{n^2x^2} = \frac{40}{100}$

$\Rightarrow \quad \frac{n-1}{2n} = \frac{2}{5}$

$\Rightarrow \quad 5n - 5 = 4n$

$\Rightarrow \quad n = 5$ and $x = 2$

Hence, the expansion is $(1 + 2)^5$.

16. Given expansion is $(1 - 3x + 7x^2)(1 - x)^{16}$

$= (1 - 3x + 7x^2)({}^{16}C_0 1^{16} - {}^{16}C_1 1^{15}x^1 + {}^{16}C_2 1^{14}x^2 - \ldots + {}^{16}C_{16}x^{16})$

$= (1 - 3x + 7x^2)(1 - 16x + 120x^2 - \ldots + x^{16})$

$\therefore$ Coefficient of $x = -3 - 16 = -19$.

17. $7^{400} = (7^4)^{100} = (2401)^{100} = (1 + 2400)^{100}$

$= {}^{100}C_0 + {}^{100}C_1(2400) + {}^{100}C_2(2400)^2 + \ldots + {}^{100}C_{100}(2400)^{100}$

$= 1 + 100\,k$, where $k \in N$

Hence the last two digits are 01.

OR

We have $3^{3n} = 27^n = (1 + 26)^n$

$= {}^nC_0(1)^n + {}^nC_1(1)^{n-1}(26) + {}^nC_2(1)^{n-2}(26)^2 + {}^nC_3(1)^{n-3}(26)^3 + \ldots + {}^nC_n(26)^n$

$= 1 + 26n + {}^nC_2(26)^2 + {}^nC_3(26)^3 + \ldots + {}^nC_n(26)^n$

$\therefore \quad 3^{3n} - 26n - 1 = (26)^2[{}^nC_2 + {}^nC_3(26) + \ldots + {}^nC_n(26)^{n-2}]$

$= 676[{}^nC_2 + {}^nC_3(26) + \ldots + {}^nC_n(26)^{n-2}]$

which shows that $3^{3n} - 26n - 1$ is divisible by 676.

18. Clearly, $(2x - 1)^4 + 4(2x - 1)^3(3 - 2x) + 6(2x - 1)^2(3 - 2x)^2 + 4(2x - 1)^1(3 - 2x)^3 + (3 - 2x)^4$

$= {}^4C_0(2x - 1)^{4-0} + {}^4C_1(2x - 1)^{4-1}(3 - 2x)^1 + {}^4C_2(2x - 1)^{4-2}(3 - 2x)^2 + {}^4C_3(2x - 1)^{4-3}(3 - 2x)^3 + {}^4C_4(2x - 1)^{4-4}(3 - 2x)^4$

$= [(2x - 1) + (3 - 2x)]^4$

$= 2^4 = 16$.

19. $(1.02)^6 = (1 + 0.02)^6$

$= {}^6C_0 + {}^6C_1(0.02)^1 + {}^6C_2(0.02)^2 + {}^6C_3(0.02)^3 + {}^6C_4(0.02)^4 + {}^6C_5(0.02)^5 + {}^6C_6(0.02)^6$

$= 1 + 6 \cdot (0.02) + 15(0.0004) + 20(0.000008) + 15(0.00000016) + 6(0.0000000032) + 0.000000000064$

$= 1 + 0.12 + 0.006 + 0.00016 + 0.0000024 + \ldots$

$= 1 + 0.126 + 0.00016$

(Ignoring other terms, as answer is required to correct upto four places of decimals)

$= 1.12616$.

20. (i) Consider, L.H.S. $= C_1 + 2 \cdot C_2 + 3 \cdot C_3 + 4 \cdot C_4 + \ldots + n \cdot C_n$

$= n + 2 \cdot \frac{n(n-1)}{2} + 3 \cdot \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} + \ldots + n \cdot 1$

$= n + n(n - 1) + \frac{n(n-1)(n-2)}{2 \cdot 1} + \ldots + n$

$= n\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots + 1\right]$

$= n[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1}]$

$= n[2^{n-1}]$

= R.H.S.

(ii) Consider, L.H.S. $= {}^{2n}C_0 - 4 \cdot {}^{2n}C_1 + 4^2 \cdot {}^{2n}C_2 - \ldots + (-1)^{2n} 4^{2n}\, {}^{2n}C_{2n}$

$= {}^{2n}C_0 - {}^{2n}C_1 4^1 + {}^{2n}C_2 4^2 - \ldots + {}^{2n}C_{2n}(-1)^{2n}4^{2n}$

$= (1 - 4)^{2n} \qquad [\because (1 - x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 + \ldots + (-1)^n\, {}^nC_n x^n]$

$= (-3)^{2n} = 9^n$.

OR

(i) Let $y = 3\sqrt{1-x^2}$.

$$(2x^2 - y)^4 = {}^4C_0(2x^2)^4 - {}^4C_1(2x^2)^3y + {}^4C_2(2x^2)^2(y)^2 - {}^4C_3(2x^2)y^3 + {}^4C_4y^4$$

and $$(2x^2 + y)^4 = {}^4C_0(2x^2)^4 + {}^4C_1(2x^2)^3y + {}^4C_2(2x^2)^2y^2 + {}^4C_3(2x^2)y^3 + {}^4C_4y^4$$

$\therefore$ $(2x^2 - y)^4 + (2x^2 + y)^4$

$= 2[{}^4C_0(2x^2)^4 + {}^4C_2(2x^2)^2y^2 + {}^4C_4y^4]$

$= 2[16x^8 + 6 \cdot 4 \cdot x^4y^2 + y^4]$

$= 32x^8 + 48x^4y^2 + 2y^4$

$= 32x^8 + 48x^4[9(1 - x^2)] + 2 \times 81(1 - x^2)^2$

$= 32x^8 + 432x^4 - 432x^6 + 162(1 + x^4 - 2x^2)$

$= 32x^8 + 432x^4 - 432x^6 + 162 + 162x^4 - 324x^2$

$= 32x^8 - 432x^6 + 594x^4 - 324x^2 + 162.$

(ii) Let $y = x + x^2$

$\Rightarrow$ $(1 + x + x^2)^3 = (1 + y)^3 = {}^3C_0 + {}^3C_1y + {}^3C_2y^2 + {}^3C_3y^3$

$= 1 + 3y + 3y^2 + y^3 = 1 + 3(x + x^2) + 3(x + x^2)^2 + (x + x^2)^3$

$= 1 + 3(x + x^2) + 3(x^2 + 2x^3 + x^4) + \{{}^3C_0x^3(x^2)^0 + {}^3C_1x^{3-1}(x^2)^1 + {}^3C_2x^{3-2}(x^2)^2 + {}^3C_3x^0(x^2)^3\}$

$= 1 + 3(x + x^2) + 3(x^2 + 2x^3 + x^4) + (x^3 + 3x^4 + 3x^5 + x^6)$

$= x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1.$

21. We have, $$\left(\frac{1}{x} + x^{\log_{10}x}\right)^5 = {}^5C_0\left(\frac{1}{x}\right)^5 + {}^5C_1\left(\frac{1}{x}\right)^4\left(x^{\log_{10}x}\right) + {}^5C_2\left(\frac{1}{x}\right)^3\left(x^{\log_{10}x}\right)^2 + \ldots$$

Given, $T_3 = 1000$

$\Rightarrow$ $${}^5C_2\left(\frac{1}{x}\right)^3(x^{\log_{10}x})^2 = 1000$$

$\Rightarrow$ $10(x^{\log_{10}x})^2 \times x^{-3} = 1000$

$\Rightarrow$ $x^{2(\log_{10}x)} \times x^{-3} = 100$

$\Rightarrow$ $x^{2(\log_{10}x)-3} = 10^2$

$\Rightarrow$ $2\log_{10}x - 3 = \log_x 10^2$

$\Rightarrow$ $$2\log_{10}x - 3 = \frac{2}{\log_{10}x}$$

$\Rightarrow$ $2y - 3 = \frac{2}{y}$, where $y = \log_{10}x$

$\Rightarrow$ $2y^2 - 3y - 2 = 0$

$\Rightarrow$ $(2y + 1)(y - 2) = 0$

$\Rightarrow$ $y = 2$ or $y = -\frac{1}{2}$

$\Rightarrow$ $\log_{10}x = 2$ or $\log_{10}x = -\frac{1}{2}$

$\Rightarrow$ $x = 10^2 = 100$ or, $x = 10^{-1/2} = \frac{1}{\sqrt{10}}$.

22. (i) The coefficients of the first three terms of $\left(x - \frac{3}{x^2}\right)^m$ are ${}^mC_0$, $(-3)\,{}^mC_1$ and $9\,{}^mC_2$.

By the given condition, we have

$${}^mC_0 - 3\,{}^mC_1 + 9\,{}^mC_2 = 559 \Rightarrow 1 - 3m + \frac{9m(m-1)}{2} = 559$$

$\Rightarrow$ $9m^2 - 15m - 1116 = 0$

$\Rightarrow$ $3m^2 - 5m - 372 = 0$

$\Rightarrow$ $(m - 12)(3m + 31) = 0$

$\Rightarrow$ $m = 12$ (m being a natural number)

(ii) Consider $(1 - x)^2(2 + x)^5 = (1 + x^2 - 2x)(2 + x)^5$

$= (2 + x)^5 + x^2(2 + x)^5 - 2x(2 + x)^5$

Now, coefficient of x^4 in $(1 - x)^2(2 + x)^5$ = coefficient of x^4 in $(2 + x)^5$ + coefficient of x^2 in $(2 + x)^5$ – 2 coefficient of x^3 in $(2 + x)^5$

$\therefore$ Coefficient of $x^4 = {}^5C_4\,2 + {}^5C_2\,2^3 - 2\,{}^5C_3\,2^2$

$= 5 \cdot 2 + 10 \cdot 8 - 10 \cdot 8 = 10.$

23. Coefficient of 5th, 6th and 7th terms in the expansion of $(1 + x)^n$ are ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ respectively.

Now, according to given condition, we have

${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P.

$\because$ $2 \cdot {}^nC_5 = {}^nC_4 + {}^nC_6$

$\Rightarrow$ $$\frac{2 \cdot n!}{(n-5)!5!} = \frac{n!}{4!(n-4)!} + \frac{n!}{6!(n-6)!}$$

$\Rightarrow$ $$\frac{2}{(n-5) \cdot 5} = \frac{1}{(n-4)(n-5)} + \frac{1}{6 \cdot 5}$$

$\Rightarrow$ $$-\frac{1}{30} = \frac{1}{(n-4)(n-5)} - \frac{2}{5(n-5)}$$

$\Rightarrow$ $$-\frac{1}{30} = \frac{5 - 2(n-4)}{5(n-4)(n-5)}$$

$\Rightarrow$ $$-\frac{1}{6} = \frac{5 - 2n + 8}{n^2 - 9n + 20}$$

$\Rightarrow$ $-n^2 + 9n - 20 = 30 - 12n + 48$

$\Rightarrow$ $n^2 - 21n + 98 = 0 \Rightarrow n^2 - 14n - 7n + 98 = 0$

$\Rightarrow$ $n(n - 14) - 7(n - 14) = 0$

$\Rightarrow$ $(n - 14)(n - 7) = 0$

$\Rightarrow$ $n = 14$ or 7.