Binomial Theorem

CHAPTER 8

NCERT FOCUS

ANSWERS

EXERCISE - 8.1

1. We have, $(1-2x)^5 = [1+(-2x)]^5$
$= {}^5C_0 + {}^5C_1(-2x) + {}^5C_2(-2x)^2 + {}^5C_3(-2x)^3 + {}^5C_4(-2x)^4 + {}^5C_5(-2x)^5$
$= 1 + 5(-2x) + 10(4x^2) + 10(-8x^3) + 5(16x^4) + (-32x^5)$
$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

2. We have, $\left(\frac{2}{x}-\frac{x}{2}\right)^5 = \left[\frac{2}{x}+\left(-\frac{x}{2}\right)\right]^5$

$$= {}^5C_0\left[\frac{2}{x}\right]^5 + {}^5C_1\left[\frac{2}{x}\right]^4\left[\frac{-x}{2}\right] + {}^5C_2\left[\frac{2}{x}\right]^3\left[\frac{-x}{2}\right]^2 + {}^5C_3\left[\frac{2}{x}\right]^2\left[\frac{-x}{2}\right]^3 + {}^5C_4\left[\frac{2}{x}\right]\left[\frac{-x}{2}\right]^4 + {}^5C_5\left[\frac{-x}{2}\right]^5$$

$$= 1\left[\frac{32}{x^5}\right] + 5\left(\frac{8}{x^3}\right)(-1) + 10\left[\frac{2}{x}\right] + 10\left[\frac{-x}{2}\right] + 5\left[\frac{x^3}{8}\right] + \left[\frac{-x^5}{32}\right]$$

$$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5x^3}{8} - \frac{x^5}{32}$$

3. We have, $(2x-3)^6 = [2x+(-3)]^6$
$= {}^6C_0(2x)^6 + {}^6C_1(2x)^5(-3) + {}^6C_2(2x)^4(-3)^2 + {}^6C_3(2x)^3(-3)^3 + {}^6C_4(2x)^2(-3)^4 + {}^6C_5(2x)(-3)^5 + {}^6C_6(-3)^6$
$= 1(64x^6) + 6(32x^5)(-3) + 15(16x^4)(9) + 20(8x^3)(-27)$
$+ 15(4x^2)(81) + 6(2x)(-243) + 1(729)$
$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729.$

4. We have, $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

$$= {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5$$

$$= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$$

5. We have, $\left(x+\frac{1}{x}\right)^6$

$$= {}^6C_0(x)^6 + {}^6C_1(x)^5\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2 + {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$

$$= x^6 + 6x^4 + 15x^2 + 20 + 15\left(\frac{1}{x^2}\right) + 6\left(\frac{1}{x^4}\right) + \frac{1}{x^6}$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}.$$

6. We have, $96 = 100 - 4$
$\therefore$ $(96)^3 = (100-4)^3$
$= {}^3C_0(100)^3 + {}^3C_1(100)^2(-4) + {}^3C_2(100)(-4)^2 + {}^3C_3(-4)^3$
$= (100)^3 + 3(100)^2(-4) + 3(100)(16) + (-64)$
$= 1000000 - 120000 + 4800 - 64 = 884736$

7. We have, $102 = 100 + 2$
$\therefore$ $(102)^5 = (100+2)^5$
$= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$
$= (100)^5 + 5(100)^4(2) + 10(100)^3(4) + 10(100)^2(8) + 5(100)(16) + 1(32)$
$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$
$= 11040808032.$

8. We have, $101 = 100 + 1$
$\therefore$ $(101)^4 = (100+1)^4$
$= {}^4C_0(100)^4 + {}^4C_1(100)^3 + {}^4C_2(100)^2 + {}^4C_3(100) + {}^4C_4(100)^0$
$= 100000000 + 4(1000000) + 6(10000) + 400 + 1$
$= 100000000 + 4000000 + 60000 + 400 + 1 = 104060401$

9. We have, $99 = 100 - 1$
$\therefore$ $(99)^5 = (100-1)^5 = [100+(-1)]^5$
$= {}^5C_0(100)^5 + {}^5C_1(100)^4(-1) + {}^5C_2(100)^3(-1)^2 + {}^5C_3(100)^2(-1)^3 + {}^5C_4(100)(-1)^4 + {}^5C_5(-1)^5$
$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$
$= 9509900499.$

10. Splitting 1.1 and using binomial theorem to write the first few terms We have, $(1.1)^{10000} = (1+0.1)^{10000}$
$= {}^{10000}C_0 + {}^{10000}C_1(0.1) + {}^{10000}C_2(0.1)^2 + \ldots + {}^{10000}C_{10000}(0.1)^{10000}$

$$= 1 + 10000 \times \frac{1}{10} + \text{ some positive terms}$$

$= 1 + 1000$ + some positive terms > 1000.
Hence, $(1.1)^{10000}$ is larger than 1000.

11. By binomial theorem, we have
$(a+b)^4 - (a-b)^4 = [{}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4] - [{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4]$
$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 - a^4 + 4a^3b - 6a^2b^2 + 4ab^3 - b^4$
$\therefore$ $(a+b)^4 - (a-b)^4 = 8a^3b + 8ab^3 = 8ab(a^2+b^2)$ ... (i)
Substituting $a=\sqrt{3}$ & $b=\sqrt{2}$ in (i), we get

$$(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4 = 8\sqrt{3}\sqrt{2}\left[(\sqrt{3})^2+(\sqrt{2})^2\right]$$

$$= 8\sqrt{6}[3+2] = 40\sqrt{6}.$$

12. By using binomial theorem, we have

$(x+1)^6 + (x-1)^6 = [{}^6C_0\,x^6 + {}^6C_1\,x^5 + {}^6C_2x^4 + {}^6C_3\,x^3 + {}^6C_4\,x^2 + {}^6C_5\,x + {}^6C_6] + [{}^6C_0\,x^6 - {}^6C_1\,x^5 + {}^6C_2\,x^4 - {}^6C_3\,x^3 + {}^6C_4\,x^2 - {}^6C_5x + {}^6C_6]$

$= x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1 + x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$

$\therefore\quad (x+1)^6 + (x-1)^6 = 2x^6 + 30x^4 + 30x^2 + 2$ ... (i)

Substituting $x = \sqrt{2}$, in (i), we get

$(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2(\sqrt{2})^6 + 30(\sqrt{2})^4 + 30(\sqrt{2})^2 + 2$

$= 16 + 120 + 60 + 2 = 198.$

13. We have to prove that $9^{n+1} - 8n - 9 = 64k$

Consider, $9^{n+1} - 8n - 9 = (8+1)^{n+1} - 8n - 9$ [put $9 = 8 + 1$]

$= [{}^{n+1}C_0\,8^{n+1} + \ldots\ldots + {}^{n+1}C_{n-2}8^3 + {}^{n+1}C_{n-1}\,8^2 + {}^{n+1}C_n\,8 + {}^{n+1}C_{n+1}] - 8n - 9$

$= {}^{n+1}C_0\,8^{n+1} + \ldots\ldots + {}^{n+1}C_{n-2}\,8^3 + {}^{n+1}C_{n-1}\,8^2 + (n+1)8 + 1 - 8n - 9$

$= {}^{n+1}C_0\,8^{n+1} + \ldots\ldots + {}^{n+1}C_{n-2}\,8^3 + {}^{n+1}C_{n-1}\,8^2 + 8n + 8 + 1 - 8n - 9$

$= {}^{n+1}C_0\,8^{n+1} + \ldots\ldots + {}^{n+1}C_{n-2}\,8^3 + {}^{n+1}C_{n-1}\,8^2$

$= 8^2\,[{}^{n+1}C_0\,8^{n-1} + \ldots\ldots + {}^{n+1}C_{n-2}\,8 + {}^{n+1}C_{n-1}]$

$= 64k$ [where, $k = {}^{n+1}C_0\,8^{n-1} + \ldots\ldots + {}^{n+1}C_{n-1}$]

Hence, $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

14. We have,

$\sum_{r=0}^{n} 3^r\,{}^nC_r = {}^nC_0 3^0 + {}^nC_1 3^1 + {}^nC_2 3^2 + \ldots\ldots + {}^nC_n 3^n$

$= (1+3)^n$ [$\because\ {}^nC_0\,a^0 + {}^nC_1 a + \ldots\ldots + {}^nC_n a^n = (1+a)^n$]

$= 4^n$. Hence proved.

NCERT MISCELLANEOUS EXERCISE

1. We have, $(3 + ax)^9 = {}^9C_0\,3^9\,(ax)^0 + {}^9C_1\,3^8\,(ax) + {}^9C_2\,3^7\,(ax)^2 + {}^9C_3\,3^6\,(ax)^3 + \ldots$

We are given coefficient of x^2 = coefficient of x^3

$\therefore\quad 36\,(3)^7\,a^2 = 84\,(3)^6\,a^3$

$\Rightarrow\quad \frac{a^3}{a^2} = \frac{36\,(3)^7}{84\,(3)^6} = \frac{9}{7} \Rightarrow a = \frac{9}{7}.$

2. We first expand each of the factors of the given product using binomial theorem. We have

$(1+2x)^6 = {}^6C_0 + {}^6C_1\,(2x) + {}^6C_2\,(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 + {}^6C_5(2x)^5 + {}^6C_6\,(2x)^6.$

$= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$

and $(1-x)^7 = {}^7C_0 + {}^7C_1\,(-x) + {}^7C_2\,(-x)^2 + {}^7C_3\,(-x)^3 + {}^7C_4\,(-x)^4 + {}^7C_5\,(-x)^5 + {}^7C_6(-x)^6 + {}^7C_7(-x)^7$

$= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7.$

Thus $(1+2x)^6\,(1-x)^7$

$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6) \times (1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7)$

We write only those terms which involves x^5. This can be done if we note, that $x^r \cdot x^{5-r} = x^5$.

The terms containing x^5 are

$1\,(-21x^5) + 12x\,(35x^4) + 60x^2\,(-35x^3) + 160x^3\,(21x^2) + 240x^4\,(-7x) + 192x^5\,(1)$

$= -21x^5 + 420x^5 - 2100x^5 + 3360x^5 - 1680x^5 + 192x^5$

$= 171x^5$

Thus, the coefficients of x^5 in the given product is 171.

3. We can write $a^n = (a - b + b)^n$

Then $a^n = [(a-b) + b]^n$

$= {}^nC_0\,(a-b)^n + {}^nC_1\,(a-b)^{n-1}\,b + \ldots. + {}^nC_{n-1}\,(a-b)b^{n-1} + {}^nC_n\,b^n$

$\Rightarrow\quad a^n - b^n = {}^nC_0\,(a-b)^n + {}^nC_1\,(a-b)^{n-1}\,b + \ldots. + {}^nC_{n-1}(a-b)\,b^{n-1} + b^n - b^n$

$= (a-b)\,[{}^nC_0\,(a-b)^{n-1} + {}^nC_1\,(a-b)^{n-2}\,b + \ldots. + {}^nC_{n-1}\,b^{n-1}]$

$= (a-b)$ (some integer)

[${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_{n-1}$, are integers & also all non negative powers of $a - b$ and b]

Hence, $a - b$ is a factor of $a^n - b^n$.

4. We have

$(\sqrt{3}+\sqrt{2})^6 = {}^6C_0(\sqrt{3})^6 + {}^6C_1(\sqrt{3})^5(\sqrt{2}) + {}^6C_2(\sqrt{3})^4(\sqrt{2})^2 + {}^6C_3(\sqrt{3})^3(\sqrt{2})^3 + {}^6C_4(\sqrt{3})^2(\sqrt{2})^4 + {}^6C_5(\sqrt{3})(\sqrt{2})^5 + {}^6C_6(\sqrt{2})^6$ ...(i)

and $(\sqrt{3}-\sqrt{2})^6 = {}^6C_0(\sqrt{3})^6 - {}^6C_1(\sqrt{3})^5(\sqrt{2}) + {}^6C_2(\sqrt{3})^4(\sqrt{2})^2 - {}^6C_3(\sqrt{3})^3(\sqrt{2})^3 + {}^6C_4(\sqrt{3})^2(\sqrt{2})^4 - {}^6C_5(\sqrt{3})(\sqrt{2})^5 + {}^6C_6(\sqrt{2})^6$ ...(ii)

Subtracting (ii) from (i), we get $(\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6$

$= 2\,[{}^6C_1(\sqrt{3})^5(\sqrt{2}) + {}^6C_3(\sqrt{3})^3(\sqrt{2})^3 + {}^6C_5(\sqrt{3})(\sqrt{2})^5]$

$= 2\,[6(9\sqrt{3})(\sqrt{2}) + 20(3\sqrt{3})(2\sqrt{2}) + 6(\sqrt{3})(4\sqrt{2})]$

$= 2\,[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}] = 2(198\sqrt{6}) = 396\sqrt{6}$

5. We have, $\left(a^2+\sqrt{a^2-1}\right)^4 = {}^4C_0(a^2)^4 + {}^4C_1\,(a^2)^3\left(\sqrt{a^2-1}\right) + {}^4C_2\,(a^2)^2\left(\sqrt{a^2-1}\right)^2 + {}^4C_3\,(a^2)\left(\sqrt{a^2-1}\right)^3 + {}^4C_4\left(\sqrt{a^2-1}\right)^4$ ... (i)

and $\left(a^2-\sqrt{a^2-1}\right)^4$

$= {}^4C_0(a^2)^4 - {}^4C_1\,(a^2)^3\left(\sqrt{a^2-1}\right) + {}^4C_2\,(a^2)^2\left(\sqrt{a^2-1}\right)^2 - {}^4C_3(a^2)\left(\sqrt{a^2-1}\right)^3 + {}^4C_4\left(\sqrt{a^2-1}\right)^4$ ... (ii)

Adding (i) and (ii), we get

$$\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4$$

$$=2\left[{}^4C_0\,(a^2)^4+{}^4C_2\,(a^2)^2\left(\sqrt{a^2-1}\right)^2+{}^4C_4\left(\sqrt{a^2-1}\right)^4\right]$$

$= 2[a^8 + 6a^4(a^2 - 1) + (a^2 - 1)^2] = 2\,[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1]$

$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2.$

6. We have, $0.99 = (1 - 0.01)$

$\therefore$ $(0.99)^5 = (1 - 0.01)^5$

$= {}^5C_0 + {}^5C_1\,(-0.01) + {}^5C_2(-0.01)^2 + {}^5C_3(-0.01)^3 + {}^5C_4\,(-0.01)^4 + {}^5C_5(-0.01)^5$

$= 1 + 5\,(-0.01) + 10\,(0.0001) + {}^5C_3\,(-0.01)^3 + {}^5C_4\,(-0.01)^4 + {}^5C_5\,(-0.01)^5$

$= 0.951 + {}^5C_3\,(-0.01)^3 + {}^5C_4\,(-0.01)^4 + {}^5C_5 \times (-0.01)^5$

Hence, $(0.99)^5$ is nearly equal to 0.951.

7. Let $\frac{x}{2}-\frac{2}{x}=y$

$$\therefore\ \left(1+\frac{x}{2}-\frac{2}{x}\right)^4=(1+y)^4$$

$= {}^4C_0 + {}^4C_1y + {}^4C_2y^2 + {}^4C_3y^3 + {}^4C_4y^4$

$$={}^4C_0+{}^4C_1\left[\frac{x}{2}-\frac{2}{x}\right]+{}^4C_2\left[\frac{x}{2}-\frac{2}{x}\right]^2+{}^4C_3\left[\frac{x}{2}-\frac{2}{x}\right]^3+{}^4C_4\left[\frac{x}{2}-\frac{2}{x}\right]^4$$

$$=1+4\left[\frac{x}{2}-\frac{2}{x}\right]+6\left[{}^2C_0\left(\frac{x}{2}\right)^2+{}^2C_1\left(\frac{x}{2}\right)\left(\frac{-2}{x}\right)+{}^2C_2\left(\frac{-2}{x}\right)^2\right]+4\left[{}^3C_0\left(\frac{x}{2}\right)^3+{}^3C_1\left(\frac{x}{2}\right)^2\left(\frac{-2}{x}\right)+{}^3C_2\left(\frac{x}{2}\right)\left(\frac{-2}{x}\right)^2+{}^3C_3\left(\frac{-2}{x}\right)^3\right]$$

$$+\left[{}^4C_0\left(\frac{x}{2}\right)^4+{}^4C_1\left(\frac{x}{2}\right)^3\left(\frac{-2}{x}\right)+{}^4C_2\left(\frac{x}{2}\right)^2\left(\frac{-2}{x}\right)^2+{}^4C_3\left(\frac{x}{2}\right)\left(\frac{-2}{x}\right)^3+{}^4C_4\left(\frac{-2}{x}\right)^4\right]$$

$$=1+2x-\frac{8}{x}+6\left[\frac{x^2}{4}-2+\frac{4}{x^2}\right]+4\left[\frac{x^3}{8}-\frac{3x}{2}+\frac{6}{x}-\frac{8}{x^3}\right]+\left[\frac{x^4}{16}-x^2+6-\frac{16}{x^2}+\frac{16}{x^4}\right]$$

$$=1+2x-\frac{8}{x}+\frac{3}{2}x^2-12+\frac{24}{x^2}+\frac{x^3}{2}-6x+\frac{24}{x}-\frac{32}{x^3}+\frac{x^4}{16}-x^2+6-\frac{16}{x^2}+\frac{16}{x^4}$$

$$=\frac{16}{x}+\frac{8}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-5.$$

8. Let $3x^2 - 2ax = y$

Then $[3x^2 - 2ax + 3a^2]^3 = [y + 3a^2]^3$

$= {}^3C_0y^3 + {}^3C_1y^2(3a^2) + {}^3C_2y\,(3a^2)^2 + {}^3C_3\,(3a^2)^3$

$= (3x^2 - 2ax)^3 + 3(3x^2 - 2ax)^2\,(3a^2) + 3(3x^2 - 2ax)\,(9a^4) + (27a^6)$

$= [{}^3C_0(3x^2)^3 + {}^3C_1(3x^2)^2\,(-2ax) + {}^3C_2\,(3x^2)\,(-2ax)^2 + {}^3C_3(-2ax)^3] + 3[{}^2C_0(3x^2)^2 + {}^2C_1(3x^2)(-2ax) + {}^2C_2(-2ax)^2](3a^2) + 3(27x^2a^4 - 18a^5x) + 27a^6$

$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 - 108a^3x^3 + 36a^4x^2 + 81a^4x^2 - 54a^5x + 27a^6$

$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$