Sequences and Series

CHAPTER 9

EXAM DRILL

ANSWERS

1. (c) : We have, $a_n = \dfrac{n(n-4)}{n^2+1}$ ...(i)

Put $n = 9$ in (i), we get $a_9 = \dfrac{9\times 5}{(9)^2+1} = \dfrac{45}{82}$

2. (c) : $1 + \sin\theta + \sin^2\theta + \ldots$ upto $\infty = 4+2\sqrt{3}$

Since, L.H.S is an infinite G.P. with common ratio $\sin\theta$.

$\therefore \quad \dfrac{1}{1-\sin\theta} = 4+2\sqrt{3}$ $\left[\text{Since, } S_\infty = \dfrac{a}{1-r}\right]$

$\Rightarrow \quad 1 - \sin\theta = \dfrac{1}{4+2\sqrt{3}} \times \dfrac{4-2\sqrt{3}}{4-2\sqrt{3}} = \dfrac{4-2\sqrt{3}}{16-12} = \dfrac{2-\sqrt{3}}{2}$

$\Rightarrow \quad 2 - 2\sin\theta = 2-\sqrt{3} \Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta = \dfrac{\pi}{3}$

3. (b) : We have x, $2y$ and $3z$ are in A.P.

$\Rightarrow \quad 2y = \dfrac{x+3z}{2}$

$\Rightarrow \quad 4y = x + 3z$ ...(i)

and x, y, z are in G.P.

$\Rightarrow \quad \dfrac{y}{x} = \dfrac{z}{y} = r$

$\Rightarrow \quad y = rx$ and $z = ry = r^2x$

Putting these values in (i), we get

$4rx = x + 3r^2x$

$\Rightarrow \quad 4r = 1 + 3r^2 \Rightarrow 3r^2 - 4r + 1 = 0$

$\Rightarrow \quad (3r-1)(r-1) = 0 \Rightarrow r = 1, \dfrac{1}{3}.$

$\therefore \quad r = \dfrac{1}{3}$ [Reject $r = 1$, as x, y, z are distinct]

4. (c) : Given series $3, 1, \dfrac{1}{3}, \ldots$ forms a G.P.

where first term $(a) = 3$ and common ratio $(r) = \dfrac{1}{3}$

$\therefore$ Sixth term, $a_6 = ar^5$

$= (3)\left(\dfrac{1}{3}\right)^5 = \left(\dfrac{1}{3}\right)^4 = \dfrac{1}{81}$

5. (d) : Given, arithmetic mean of x and y is 3

$\Rightarrow \quad \dfrac{x+y}{2} = 3 \quad \Rightarrow \quad x + y = 6$ ...(i)

and geometric mean of x and y is 1

$\Rightarrow \quad \sqrt{xy} = 1 \quad \Rightarrow \quad xy = 1$ ...(ii)

Squaring (i) on both sides, we get

$(x+y)^2 = (6)^2 \quad \Rightarrow \quad x^2 + y^2 + 2xy = 36$

$\Rightarrow \quad x^2 + y^2 + 2 = 36$ [Using (ii)]

$\Rightarrow \quad x^2 + y^2 = 34$

6. (b) : $a_1 = 1$; $a_2 = 2a_1 = 2$;

$a_3 = 3a_2 = 3 \times 2 = 6$; $a_4 = 4a_3 = 4 \times 6 = 24$;

$a_5 = 5a_4 = 5 \times 24 = 120$

7. (c) : Let the first term is a and common ratio is r

$\therefore \quad a_3 = ar^2 = 5$

$\therefore$ Product of its first 5 terms $= a \times ar \times ar^2 \times ar^3 \times ar^4$

$= (ar^2)^5 = 5^5 = 3125$

8. Geometric mean of a^3b and ab^3

$= \sqrt{a^3b \times ab^3} = \sqrt{a^4b^4} = a^2b^2$

9. The given series is a G.P. in which first term, $a = 2$, common ratio, $r = 3$ and last term, $ar^{n-1} = = 4374$.

$\therefore$ Required sum, $S_n = \dfrac{a(r^n-1)}{r-1} = \dfrac{ar^n - a}{r-1} = \dfrac{ar^{n-1}\cdot r - a}{r-1}$

$= \dfrac{4374\times 3 - 2}{3-1} = 6560$

10. Given, $\log_x a$, $a^{x/2}$ and $\log_b x$ are in G.P.

$\Rightarrow \quad (a^{x/2})^2 = \log_x a \times \log_b x$

$\Rightarrow \quad a^x = \log_b a$

$\Rightarrow \quad \log_a a^x = \log_a \log_b a$

$\Rightarrow \quad x = \log_a \log_b a$

11. Given a, b, c are in G.P. $\Rightarrow b^2 = ac$

L.H.S $= a(b^2 + c^2) = a\,(ac + c^2) = a^2c + ac^2 = c(a^2 + ac)$

$= c\,(a^2 + b^2) =$ R.H.S.

12. Let the first three consecutive terms of G.P. be $\dfrac{a}{r}, a, ar.$

According to the question,

$\dfrac{a}{r} \times a \times ar = 27 \Rightarrow a^3 = 27 \Rightarrow a = 3$

13. (i) (c) : If n geometric mean $g_1, g_2, \ldots, g_n$ are to be inserted between two positive real numbers x and y, then $x, g_1, g_2 \ldots g_n, y$ are in G.P.

Then, $g_1 = x^r$, $g_2 = xr^2 \ldots\ldots g_n = xr^n$

So, $y = xr^{n+1}$

$\Rightarrow \quad r = \left(\dfrac{y}{x}\right)^{\frac{1}{(n+1)}}$

Now, n^{th} geometric mean $g_n = x\left(\dfrac{y}{x}\right)^{\frac{n}{n+1}}$

(ii) (a) : We have $x, g_1, g_2, g_3 \ldots\ldots g_n, \frac{1}{x}$

$$g_1 \cdot g_2 \cdot g_3 \cdot \ldots\ldots g_n = \left(x \times \frac{1}{x}\right)^{n/2} = 1$$

(iii) (b) : Given that Ram is 12 years older than Ankit and arithmetic mean of their ages exceed the geometric mean of their ages by 2

$\therefore \quad x - y = 12$ ....(1)

and A.M. – G.M = 2 ....(2)

$\Rightarrow \frac{x+y}{2} - \sqrt{xy} = 2$

$\Rightarrow \quad = x + y - 2\sqrt{xy} = 4$

$\Rightarrow \quad (\sqrt{x} - \sqrt{y})^2 = 4$

$\Rightarrow \quad \sqrt{x} - \sqrt{y} = 2$ ...(3)

Now, $x - y = 12$

$(\sqrt{x} + \sqrt{y})\ (\sqrt{x} - \sqrt{y}) = 12$

$\Rightarrow \quad (\sqrt{x} + \sqrt{y}) \times (2) = 12$

$\Rightarrow \quad \sqrt{x} + \sqrt{y} = 6$ ...(4)

By (3) and (4), $x = 16$ and $y = 4$.

(iv) (c) : Let y and x be two geometric mean such that $1, y, x, 64$, then

$y^2 = x$ ...(1) and $x^2 = 64y$ ...(2)

Put $x = y^2$ in(2), we get $y^4 = 64y$

or $y^3 = 64 \Rightarrow y = 4$

Now, by (1), $x = (4)^2 \Rightarrow x = 16$

$\therefore$ Geometric means are 4, 16.

(v) (b)

14. Here, first term, $A = \frac{1}{a^3x^3}$, common ratio,

$$r = \frac{ax}{1/a^3x^3} = a^4x^4$$

$\therefore$ 12th term, $a_{12} = Ar^{12-1} = \frac{1}{a^3x^3}\left(a^4x^4\right)^{11} = a^{41}x^{41}$

Hence, the 12th term of the given G.P = $a^{41}\,x^{41}$.

15. It is given that a, b, c are in G.P.

$\therefore \quad b^2 = ac \Rightarrow (b^2)^n = (ac)^n \Rightarrow b^{2n} = a^n c^n$

$\Rightarrow \quad \log b^{2n} = \log (a^n c^n)$

$\Rightarrow \quad \log(b^n)^2 = \log a^n + \log c^n$

$\Rightarrow \quad 2 \log b^n = \log a^n + \log c^n$

$\Rightarrow \quad \log a^n, \log b^n, \log c^n$ are in A.P.

16. Let r be the common ratio of the G.P. Then, $b = ar$ and $c = ar^2$.

Now, $a + b + c = xb$

$\Rightarrow \quad a + ar + ar^2 = xar \Rightarrow 1 + r + r^2 = xr$

$\Rightarrow \quad r^2 + (1 - x)\, r + 1 = 0$

But, r is real.

$\therefore$ Disc > 0

$\Rightarrow \quad (1 - x)^2 - 4(1)(1) > 0$

$\Rightarrow \quad 1 + x^2 - 2x - 4 > 0$

$\Rightarrow \quad x^2 - 2x - 3 > 0$

$\Rightarrow \quad (x + 1)(x - 3) > 0$

$\Rightarrow \quad x < -1$ or $x > 3$

17. Since, A.M. $\geq$ G.M.

$$\Rightarrow \frac{x^{\log y - \log z} + y^{\log z - \log x} + z^{\log x - \log y}}{3}$$

$$\geq \sqrt[3]{x^{\log y - \log z} \cdot y^{\log z - \log x} \cdot z^{\log x - \log y}} \quad \text{...(i)}$$

Since, $\log\left[x^{\log y - \log z} \cdot y^{\log z - \log x} \cdot z^{\log x - \log y}\right]$

$= (\log y - \log z) \log x + (\log z - \log x) \log y + (\log x - \log y) \log z = 0$

$\therefore \quad x^{\log y - \log z} \cdot y^{\log z - \log x} \cdot z^{\log x - \log y} = 1$

From (i),

$\therefore \quad x^{\log y - \log z} + y^{\log z - \log x} + z^{\log x - \log y} \geq 3$

$\therefore$ Minimum value is 3.

18. Let a be the first term and d be the common difference of the A.P.

$a_p = a + (p - 1)d$ ...(i) $a_q = a + (q - 1)d$ ...(ii)

$a_r = a + (r - 1)d$ ...(iii) $a_s = a + (s - 1)d$ ...(iv)

Given that a_p, a_q, a_r and a_s are in G.P.

So, $\frac{a_q}{a_p} = \frac{a_r}{a_q} = \frac{a_q - a_r}{a_p - a_q} = \frac{(q-r)d}{(p-q)d} = \frac{q-r}{p-q}$ ...(v)

Similarly, $\frac{a_r}{a_q} = \frac{a_s}{a_r} = \frac{a_r - a_s}{a_q - a_r} = \frac{(r-s)d}{(q-r)d} = \frac{r-s}{q-r}$ ...(vi)

By (v) and (vi), we get

$$\frac{q-r}{p-q} = \frac{r-s}{q-r}$$

$\therefore \quad p - q, q - r$ and $r - s$ are also in G.P.

19. Let the required numbers be a, ar, ar^2. Then,

Sum of three numbers = 52 $\Rightarrow a + ar + ar^2 = 52$

$\Rightarrow a(1 + r + r^2) = 52$ ...(i)

Sum of the product of numbers in pairs = 624

$\Rightarrow \quad a \cdot ar + ar \cdot ar^2 + a \cdot ar^2 = 624$

$\Rightarrow \quad a^2r\,(1 + r + r^2) = 624$ ...(ii)

Dividing (ii) by (i), we get

$ar = 12 \Rightarrow a = \frac{12}{r}$ ...(iii)

Putting $a = \frac{12}{r}$ in (i), we get

$$\frac{12}{r}(1 + r + r^2) = 52$$

$\Rightarrow \quad 3r^2 - 10r + 3 = 0$

$\Rightarrow \quad (3r - 1)\,(r - 3) = 0 \Rightarrow r = 1/3$ or $r = 3$

From (iii), $r = 3 \Rightarrow a = 4$ and $r = 1/3 \Rightarrow a = 36$.

Hence, the numbers are 4, 12, 36 or 36, 12, 4.

20. We have, $(m^2 + 1)x^2 - 3x + (m^2 + 1)^2 = 0$

$\therefore$ Sum of roots = $\frac{3}{m^2 + 1}$ and,

Product of roots = $(m^2 + 1)$

Now, b = Least value of the product of roots.

= Least value of $(m^2 + 1) = 1$ $[\because m^2 + 1 > 1$ for all $m]$

Also, c = Greatest value of the sum of the roots

= Greatest value of $\dfrac{3}{m^2+1}$

Clearly, $\dfrac{3}{m^2+1}$ is greatest when $m^2 + 1$ is least and the least value of $m^2 + 1$ is 1.

$\therefore \quad c = \dfrac{3}{1} = 3$

So, first term of the infinite G.P. is $b + 2 = 1 + 2 = 3$ and the common ratio is $\dfrac{2}{c} = \dfrac{2}{3}$.

Hence, the sum of the infinite G.P. is given by

$$S_\infty = \frac{3}{1-\frac{2}{3}} = 9 \qquad \left[\text{Using } S = \frac{a}{1-r}\right]$$

21. Let a and b be two given quantities. It is given that G is the geometric mean of a and b

$\therefore \quad G = \sqrt{ab} \Rightarrow G^2 = ab$

It is also given that A_1, A_2 are two arithmetic means between a and b. Therefore, a, A_1, A_2, b is an A.P. with common difference, $d = \dfrac{b-a}{3}$.

$$\therefore \quad A_1 = a + d = a + \frac{b-a}{3} = \frac{2a+b}{3},$$

$$\therefore \quad A_2 = a + 2d = a + \frac{2(b-a)}{3} = \frac{a+2b}{3}$$

$$\text{So, } 2A_1 - A_2 = 2\left(\frac{2a+b}{3}\right) - \left(\frac{a+2b}{3}\right) = a$$

$$\text{and } 2A_2 - A_1 = 2\left(\frac{a+2b}{3}\right) - \left(\frac{2a+b}{3}\right) = b$$

$\therefore \quad (2A_1 - A_2)(2A_2 - A_1) = ab$

$\Rightarrow \quad (2A_1 - A_2)(2A_2 - A_1) = G^2$

22. It is given that p, q, r are in G.P.

$\therefore \quad q^2 = pr$ ...(i)

Now, $px^2 + 2qx + r = 0$

$\Rightarrow \quad px^2 + 2\sqrt{pr}x + r = 0 \;\Rightarrow\; (\sqrt{p}x + \sqrt{r})^2 = 0$

$\Rightarrow \quad x = -\sqrt{\dfrac{r}{p}}$

It is given that $-\sqrt{\dfrac{r}{p}}$ is the common root of both equations,

so $-\sqrt{\dfrac{r}{p}}$ is also a root of the equation $dx^2 + 2ex + f = 0$

$$\therefore \quad d\frac{r}{p} - 2e\sqrt{\frac{r}{p}} + f = 0$$

$$\Rightarrow \quad \frac{d}{p} - 2e\sqrt{\frac{1}{pr}} + \frac{f}{r} = 0 \Rightarrow \frac{d}{p} - \frac{2e}{q} + \frac{f}{r} = 0$$

$$\Rightarrow \quad \frac{2e}{q} = \frac{d}{p} + \frac{f}{r} \;\Rightarrow\; \frac{d}{p}, \frac{e}{q}, \frac{f}{r} \text{ are in A.P.}$$

23. Given, $S_1, S_2, \ldots, S_n$ are sums of n terms of n G.P.'s. whose first term is 1 in each case and the common ratio are 1, 2, 3, ..., n respectively.

$\therefore \quad S_1 = 1 + 1 + \ldots\ n$ terms $= n$ $(r = 1)$

$S_2 = \dfrac{1(2^n-1)}{(2-1)} = \dfrac{2^n-1}{1}$ $(r = 2)$

$S_3 = 1 \cdot \dfrac{(3^n-1)}{(3-1)} = \dfrac{3^n-1}{2}$ $(r = 3)$

$S_4 = 1 \cdot \dfrac{(4^n-1)}{(4-1)} = \dfrac{4^n-1}{3}$ $(r = 4)$

....

$S_n = 1 \cdot \dfrac{(n^n-1)}{(n-1)}$ $(r = n)$.

Now, $S_1 + S_2 + 2S_3 + 3S_4 + \ldots + (n-1)S_n$

$$= n + (2^n - 1) + 2 \cdot \frac{(3^n-1)}{2} + 3 \cdot \frac{(4^n-1)}{3} + \ldots + (n-1)\frac{(n^n-1)}{(n-1)}$$

$= n + 2^n - 1 + 3^n - 1 + 4^n - 1 + \ldots + (n^n - 1)$

$= n + 2^n + 3^n + \ldots + n^n - (1 + 1 + 1 + \ldots\ (n-1)$ terms$)$

$= n - (n-1) + 2^n + 3^n + \ldots + n^n$

$= 1^n + 2^n + 3^n + \ldots + n^n$

Hence proved.

24. If A, d be the first term and common difference of A.P. respectively and x, R be the first term and common ratio of G.P., respectively.

Then, $A + (p-1)d = a$ ...(i)

$A + (q-1)d = b$ ...(ii)

$A + (r-1)d = c$ ...(iii)

and $a = xR^{p-1}$ ...(iv)

$b = xR^{q-1}$ ...(v)

$c = xR^{r-1}$ ...(vi)

Subtracting (ii) from (i), we get

$d(p - 1 - q + 1) = a - b$

$\Rightarrow \quad a - b = d(p - q)$ ...(vii)

Subtracting (iii) from (ii), we get

$d(q - 1 - r + 1) = b - c$

$\Rightarrow \quad b - c = d(q - r)$ ...(viii)

Subtracting (i) from (iii), we get

$d(r - 1 - p + 1) = c - a$

$\Rightarrow \quad c - a = d(r - p)$ ...(ix)

Now, $a^{b-c} \cdot b^{c-a} \cdot c^{a-b}$

$(xR^{p-1})^{d(q-r)} \cdot (xR^{q-1})^{d(r-p)} \cdot (xR^{r-1})^{d(p-q)}$

[Using (iv), (v), (vi) and (vii), (viii), (ix)]

$= x^{d(q-r+r-p+p-q)} R^{d\,(pq - pr - q + r + qr - pq - r + p + rp - rq - p + q)}$

$= x^0R^0 = 1$