Three Dimensional Geometry

CHAPTER 11

TRY YOURSELF SOLUTIONS

1. Let the *d.c.*'s of the lines be l, m, n. Then $l = \cos 90° = 0$, $m = \cos 60° = \frac{1}{2}$, $n = \cos 30° = \frac{\sqrt{3}}{2}$.

2. Direction cosines are

$$\pm\frac{1}{\sqrt{1^2+(-2)^2+2^2}}, \pm\frac{-2}{\sqrt{1^2+(-2)^2+2^2}}, \pm\frac{2}{\sqrt{1^2+(-2)^2+2^2}}$$

or $\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}$ or $\frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}$.

3. Clearly, the direction ratios of PQ are
$2 - 6, -3 -(-7), 1 - (-1)$
or $-4, 4, 2$ or $-2, 2, 1$
$\therefore$ Its direction cosines are

$$\pm\frac{-2}{\sqrt{(-2)^2+2^2+1^2}}, \pm\frac{2}{\sqrt{(-2)^2+2^2+1^2}}, \pm\frac{1}{\sqrt{(-2)^2+2^2+1^2}}$$

$$= \pm\left(-\frac{2}{3}\right), \pm\left(\frac{2}{3}\right), \pm\left(\frac{1}{3}\right)$$

Since α is acute, $\therefore l = \cos\alpha > 0$

$\therefore \quad l = -\left(-\frac{2}{3}\right) = \frac{2}{3}, m = -\frac{2}{3}, n = \frac{-1}{3}$ (Taking –ve sign)

$\therefore$ Required direction cosines are $\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}$

4. The points A, B, C will be collinear if the direction cosines of AB and BC are the same, (two parallel lines through a point).
Direction ratios of AB are $5 - 3, 1 - 2, 4 + 1$, *i.e.*, $2, -1, 5$
Direction ratios of CB are $5 + 1, 1 - 4, 4 + 11$ *i.e.*, $6, -3, 15$ or $2, -1, 5$
$\therefore$ AB and BC are parallel
But AB and BC have a common point C.
$\therefore$ AB and BC are same line
Hence, points A, B, C are collinear.

5. It is given that the line passes through the point $A(1, 2, 3)$. Therefore, the position vector of A is $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$. Also, it is parallel to the vector $\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$.
It is known that the line which passes through point $\vec{a}$ and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a real number.
$\therefore$ $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$ is the required equation of the line.

6. Let $A \equiv (3, 4, -7)$ and $B \equiv (1, -1, 6)$
Now, $\vec{a} = 3\hat{i} + 4\hat{j} - 7\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 6\hat{k}$
Equation of the line passing through $A(\vec{a})$ and $B(\vec{b})$ is
$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ or $\vec{r} = 3\hat{i} + 4\hat{j} - 7\hat{k} + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$
Equation in cartesian form is

$$\frac{x-3}{1-3} = \frac{y-4}{-1-4} = \frac{z+7}{6+7} \quad \text{or} \quad \frac{x-3}{-2} = \frac{y-4}{-5} = \frac{z+7}{13}$$

7. The given line passes through the point $(1, -2, 5)$ and is parallel to the vector $2\hat{i} + 3\hat{j} - \hat{k}$.
$\therefore$ The vector equation of the line is
$\vec{r} = \hat{i} - 2\hat{j} + 5\hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$

8. The given lines are $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ ...(i)

and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ ...(ii)

Any point on line (i) is $(4\lambda + 5, 4\lambda + 7, -5\lambda - 3)$
Any point on line (ii) is $(7\lambda' + 8, \lambda' + 4, 3\lambda' + 5)$
For lines (i) and (ii) to intersect, these points must coincide for some value of λ and λ'.
$\therefore$ $4\lambda + 5 = 7\lambda' + 8 \Rightarrow 4\lambda - 7\lambda' = 3$ ...(iii)
$4\lambda + 7 = \lambda' + 4 \Rightarrow 4\lambda - \lambda' = -3$ ...(iv)
$-5\lambda - 3 = 3\lambda' + 5 \Rightarrow 5\lambda + 3\lambda' = -8$ ...(v)
Solving these, we get $\lambda = -1; \lambda' = -1$
Clearly these values of λ and λ' satisfy (v).
$\Rightarrow$ (i) and (ii) intersect and their point of intersection is $(1, 3, 2)$.

9. Given line is $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$

Any point on the line is $A(-2 + 3\lambda, -1 + 2\lambda, 3 + 2\lambda)$...(i)
Let point A be at a distance of 5 units from the given point $P(1, 3, 3)$, then
$|AP| = 5$
$\Rightarrow \sqrt{(-2+3\lambda-1)^2 + (-1+2\lambda-3)^2 + (3+2\lambda-3)^2} = 5$
$\Rightarrow (3\lambda - 3)^2 + (2\lambda - 4)^2 + (2\lambda)^2 = 25$
$\Rightarrow 9\lambda^2 - 18\lambda + 9 + 4\lambda^2 - 16\lambda + 16 + 4\lambda^2 = 25$
$\Rightarrow 17\lambda^2 - 34\lambda = 0 \Rightarrow \lambda(\lambda - 2) = 0 \Rightarrow \lambda = 0, 2.$
From (i), when $\lambda = 0$, the points is $(-2, -1, 3)$; when $\lambda = 2$, the points is
$(-2 + 6, -1 + 4, 3 + 4)$ *i.e.*, $(4, 3, 7)$.
Hence, the required points are $(-2, -1, 3)$ and $(4, 3, 7)$.

10. The first line $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k})$ is parallel to the vector $\vec{b} = -\hat{i} + \hat{j} - 2\hat{k}$
The second line $\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k})$ is parallel to the vector $\vec{b}' = \hat{i} + 2\hat{j} - 2\hat{k}$
If θ is the angle between the lines, then

$$\cos\theta = \frac{\vec{b}\cdot\vec{b}'}{|\vec{b}||\vec{b}'|}$$

$$\cos\theta = \frac{(-1)(1)+(1)(2)+(-2)(-2)}{\sqrt{(-1)^2+1^2+(-2)^2}\sqrt{1^2+2^2+(-2)^2}}$$

$$= \frac{5}{\sqrt{6}\sqrt{9}} = \frac{5}{3\sqrt{6}} = \frac{5\sqrt{6}}{18}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{5\sqrt{6}}{18}\right)$$

11. The given equations are not in the standard form. The equations of the given lines can be re-written as

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-2}{0} \quad \text{...(i)}$$

$$\text{and } \frac{x-1}{1} = \frac{y+3/2}{3/2} = \frac{z+5}{2} \quad \text{...(ii)}$$

Let $\vec{b}_1$ and $\vec{b}_2$ be vectors parallel to (i) and (ii) respectively. Then,

$\vec{b}_1 = 3\hat{i} - 2\hat{j} + 0\hat{k}$ and $\vec{b}_2 = \hat{i} + \frac{3}{2}\hat{j} + 2\hat{k}$.

If θ is the angle between the given lines, then

$$\cos\theta = \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} = \frac{(3)(1)+(-2)(3/2)+(0)(2)}{\sqrt{3^2+(-2)^2+0^2}\sqrt{1^2+(3/2)^2+2^2}}$$

$= 0$

$\Rightarrow \theta = \pi/2.$

12. The equations of the given line are :

$$\frac{x-2}{3} = \frac{2y+1}{2} = \frac{5-z}{1}.$$

These equations can be re-written as

$$\frac{x-2}{3} = \frac{y+1/2}{1} = \frac{z-5}{-1}.$$

Clearly, direction ratios of this line are proportional to 3, 1, –1. So, the direction ratios of the parallel line are also proportional to 3, 1, –1.

The required line passes through (1, –1, 0) and its direction ratios are proportional to 3, 1, –1. So, its equation are :

$$\frac{x-1}{3} = \frac{y+1}{1} = \frac{z-0}{-1} \text{ i.e., } \frac{x-1}{3} = y+1 = -z$$

13. The required line is perpendicular to the lines which are parallel to vectors $\vec{b}_1 = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$ respectively. So, it is parallel to the vector $\vec{b} = \vec{b}_1 \times \vec{b}_2$.

$$\text{Now, } \vec{b} = \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -6\hat{i} - 3\hat{j} + 6\hat{k}$$

Thus, the required line passes through the point (2, –1, 3) and is parallel to the vector $\vec{b} = -6\hat{i} - 3\hat{j} + 6\hat{k}$. So, its vector equation is

$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(-6\hat{i} - 3\hat{j} + 6\hat{k})$ [Using $\vec{r} = \vec{a} + \lambda\vec{b}$]

$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \mu(2\hat{i} + \hat{j} - 2\hat{k})$, where $\mu = -3\lambda$.

14. The given lines are

$\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and

$\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + 2\mu(2\hat{i} - \hat{j} + \hat{k})$

Here $\vec{a}_1 = (\hat{i} + \hat{j}), \vec{a}_2 = (2\hat{i} + \hat{j} - \hat{k}), \vec{b} = (2\hat{i} - \hat{j} + \hat{k})$

Now, $(\vec{a}_2 - \vec{a}_1) = (\hat{i} - \hat{k})$

$\therefore \quad \vec{b} \times (\vec{a}_2 - \vec{a}_1) = (\hat{i} + 3\hat{j} + \hat{k})$

$\Rightarrow \quad |\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = \sqrt{1^2+3^2+1^2} = \sqrt{11}$

and $|\vec{b}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$

∴ Shortest distance between parallel lines

$$= \frac{|\vec{b} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|} = \frac{\sqrt{11}}{\sqrt{6}} = \frac{\sqrt{66}}{6} \text{ units.}$$

15. Here, $x_1 = 6$, $y_1 = 1$, $z_1 = 4$, $a_1 = 3$, $b_1 = -1$, $c_1 = 1$ and $x_2 = 0$, $y_2 = -2$, $z_2 = 2$, $a_2 = -3$, $b_2 = 2$, $c_2 = 2$

$$d = \left| \frac{\begin{vmatrix} -6 & -3 & -2 \\ 3 & -1 & 1 \\ -3 & 2 & 2 \end{vmatrix}}{\sqrt{(-2-2)^2+(-3-6)^2+(6-3)^2}} \right|$$

$$= \left| \frac{-6(-2-2)+3(6+3)-2(6-3)}{\sqrt{16+81+9}} \right| = \frac{45}{\sqrt{106}} \text{ units}$$

16. Here $\vec{a}_1 = \hat{i} + \hat{j}$, $\vec{b}_1 = 2\hat{i} - \hat{j} + \hat{k}$ and

$\vec{a}_2 = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b}_2 = 8\hat{i} - 6\hat{j} + 2\hat{k}$

Now $\vec{a}_2 - \vec{a}_1 = \hat{i} - \hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 8 & -6 & 2 \end{vmatrix}$$

$= \hat{i}(-2+6) - \hat{j}(4-8) + \hat{k}(-12+8) = 4\hat{i} + 4\hat{j} - 4\hat{k}$

$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 4 + 4 = 8$

$|\vec{b}_1 \times \vec{b}_2| = \sqrt{16+16+16} = \sqrt{48}$

$$\therefore \quad d = \left|\frac{8}{\sqrt{48}}\right| = \frac{2}{\sqrt{3}} \text{ units}$$

17. Here $\vec{a}_1 = (\hat{i} + \hat{j}), \vec{b}_1 = \hat{i} + \hat{k}$ and

$\vec{a}_2 = (\hat{i} + \hat{j}), \vec{b}_2 = -\hat{i} + \hat{j} - \hat{k}$

Now, $\vec{a}_2 - \vec{a}_1 = 0$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{vmatrix}$$

$= \hat{i}(0-1) - \hat{j}(-1+1) + \hat{k}(1-0) = -\hat{i} + \hat{k}$

Now, $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0 \Rightarrow d = 0$

∴ Lines are intersecting.