Relations and Functions

CHAPTER 1

NCERT FOCUS SOLUTIONS

EXERCISE - 1.1

1. (i) $A = \{1, 2, 3, 4, 5, 6, ..., 13, 14\}$ be the given set
$R = \{(x, y) : 3x - y = 0\}$
$\Rightarrow R = \{(1, 3), (2,6), (3, 9), (4, 12)\}$

(a) **Reflexive :**
Let $x \in A$ be any element.
Since $(x, x) \notin R \quad \therefore \quad R$ is not reflexive.

(b) **Symmetric :**
Let $x, y \in A$, $(x, y) \in R$ but $(y, x) \notin R$.
$\therefore$ R is not symmetric.

(c) **Transitive :**
Let $x, y, z \in A$, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \notin R$
For example : $(1, 3) \in R$ and $(3, 9) \in R$ but $(1, 9) \notin R$, therefore R is not transitive.
Hence R is not reflexive, not symmetric and not transitive.

(ii) N be the set of natural numbers
$R = \{(x, y) : y = x + 5$ and $x < 4\}$ be the set of natural numbers.
$\therefore$ $R = \{(1, 6) (2, 7), (3, 8)\}$

(a) **Reflexive :**
Let $x \in N$ be any element,
$(x, x) \notin R \quad \therefore \quad R$ is not reflexive.

(b) **Symmetric :**
Let $x, y \in N$, $(x, y) \in R$ but $(y, x) \notin R$
$\therefore$ R is not symmetric.

(c) **Transitive :**
Since $(1, 6) \in R$ and $(6, 7) \notin R$ and $(1, 7) \notin R$
$\therefore$ R is transitive.
Hence R is neither reflexive nor symmetric but transitive.

(iii) $A = \{1, 2, 3, 4, 5, 6\}$ be the given set
$R = \{(x, y) : y$ is divisible by x in $A\}$
$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

(a) **Reflexive :**
Let $x \in A$ be any element
Now $(x, x) \in R$ as, $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) \in R$
$\therefore$ R is reflexive.

(b) **Symmetric :**
For $x, y \in A$, $(x, y) \in R$ but $(y, x) \notin R$
i.e., $(1, 2) \in R$ but $(2, 1) \notin R$
$\therefore$ R is not symmetric.

(c) **Transitive :**
For $x, y, z \in A$, $(x, y) \in R, (y, z) \in R \Rightarrow (x, z) \in R$
Thus, R is transitive.
Hence R is reflexive and transitive but not symmetric.

(iv) Z be the set of all integers
$R = \{(x, y) : x - y$ is an integer$\}$

(a) **Reflexive :**
Let $x \in Z$ be any element, $(x, x) \in Z$ *i.e.*, $x - x = 0 \in Z$.
$\therefore$ R is reflexive

(b) **Symmetric :**
For $x, y \in Z$, $(x, y) \in R$
i.e., $x - y$ is an integer $\Rightarrow y - x$ is also an integer.
$\Rightarrow$ $(y, x) \in R$
$\therefore$ R is symmetric.

(c) **Transitive :**
Let $(x, y) \in R$ and $(y, z) \in R$
i.e., $(x - y)$ is an integer and $(y - z)$ is an integer
$\Rightarrow$ $(x - z) = (x - y + y - z) \in Z$
$\Rightarrow$ $(x, z) \in R$
Hence R is reflexive, symmetric and transitive.

(v) Relation R in the set A of human being in a town at a particular time.

(a) $R = \{(x, y) : x$ and y work at the same place$\}$

(i) **Reflexive :**
$(x, x) \in R$ because x and x work at the same place. Thus, R is a reflexive.

(ii) **Symmetric :**
If $(x, y) \in R \Rightarrow x$ and y work at the same place.
$\Rightarrow$ y and x work at the same place.
$\Rightarrow$ $(y, x) \in R$.
Thus R is symmetric.

(iii) **Transitive :**
For $(x, y) \in R$ and $(y, z) \in R$
$\Rightarrow$ x and y work at the same place and y and z work at the same place.
$\Rightarrow$ x and z work at the same place.
$\Rightarrow$ $(x, z) \in R$
Thus R is transitive.
Hence, R is reflexive, symmetric and transitive.

(b) $R = \{(x, y) : x$ and y live in the same locality$\}$

(i) **Reflexive :**
For $(x, x) \in R$ because x and x live in the same locality.

$\therefore$ R is reflexive.

(ii) **Symmetric :**

Let $(x, y) \in R \Rightarrow x$ and y live in the same locality.

$\Rightarrow$ y and x also live in the same locality.

$\Rightarrow$ $(y, x) \in R$

Thus R is symmetric.

(iii) **Transitive :**

Let $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow$ x and y live in the same locality and y and z live in the same locality.

$\Rightarrow$ x and z live in the same locality.

$\Rightarrow$ $(x, z) \in R$

Thus R is transitive.

Hence, R is reflexive, symmetric and transitive.

(c) $R = \{(x, y) : x$ is exactly 7 cm taller than $y\}$

(i) **Reflexive :**

As x is can't be exactly 7 cm taller than itself, so $(x, x) \notin R$, thus R is not reflexive.

(ii) **Symmetric :**

If x is exactly 7 cm taller than y, then y is not exactly 7 cm taller than x.

So, if $(x, y) \in R$, but $(y, x) \notin R$. So, R is not symmetric.

(iii) **Transitive :**

If x is exactly 7 cm taller than y and if y is exactly 7 cm taller than z, then it does not imply that x is exactly 7 cm taller than z. Thus R is not transitive.

Hence R is not reflexive, not symmetric and not transitive.

(d) $R = \{(x, y) : x$ is wife of $y\}$

(i) **Reflexive :**

As x can't be wife of herself, therefore, $(x, x) \notin R$, thus R is not reflexive.

(ii) **Symmetric :**

If x is wife of y, then y is not wife of x.

Thus $(x, y) \in R$, but $(y, x) \notin R$.

So, R is not symmetric.

(iii) **Transitive :**

If x is the wife of y, then y is not wife of z, *i.e.*, $(x, y) \in R$ and $(y, z) \notin R$, then $(x, z) \notin R$. So R is transitive.

Hence, R is neither reflexive, nor symmetric but transitive.

(e) $R = \{(x, y) : x$ is father of $y\}$

(i) **Reflexive :**

As x can't be father of x, so $(x, x) \notin R$, thus R is not reflexive.

(ii) **Symmetric :**

If x is father of y, then y is not father of x.

If $(x, y) \in R$, then $(y, x) \notin R$, so R is not symmetric.

(iii) **Transitive :**

Let $(x, y) \in R$ and $(y, z) \in R$, *i.e.*, $(x, z) \notin R$.

i.e., x is father of y, y is father of z, then x is not father of z *i.e.*, $(x, z) \notin R$.

So R is not transitive.

Hence R is not reflexive, not symmetric and not transitive.

2. $R = \{(a, b) : a \leq b^2\}$

Relation R is defined in the set of real numbers.

(i) **Reflexive :**

Let $a \in R \Rightarrow a \leq a^2$ which is false $\Rightarrow (a, a) \notin R$

$\therefore$ R is not reflexive.

(ii) **Symmetric :**

Let $a, b \in R$ and $(a, b) \in R \Rightarrow a \leq b^2 \not\Rightarrow b \leq a^2$

$\Rightarrow$ $(a, b) \in R$, but $(b, a) \notin R$

$\therefore$ R is not symmetric

(iii) **Transitive :**

Let $a, b, c \in R$

Consider $(a, b) \in R, (b, c) \in R$

$\Rightarrow$ $a \leq b^2$ and $b \leq c^2 \Rightarrow a \leq c^4$

$\Rightarrow$ $(a, c) \notin R$

$\therefore$ R is not transitive.

Hence R is not reflexive, not symmetric and not transitive.

3. Given $R = \{(a, b) : b = a + 1$ and $a, b \in \{1, 2, 3, 4, 5, 6\}\}$

$\Rightarrow$ $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

(i) **Reflexive :**

Consider $a \in \{1, 2, 3, 4, 5, 6\}$

But $(a, a) \notin R$. Thus R is not reflexive.

(ii) **Symmetric :**

Let $a, b \in \{1, 2, 3, 4, 5, 6\}$

Then $(a, b) \in R \Rightarrow b = a + 1 \not\Rightarrow a = b + 1$

So, $(b, a) \notin R$

$\therefore$ R is not symmetric.

(iii) **Transitive :**

Let $a, b, c \in \{1, 2, 3, 4, 5, 6\}$

Consider $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $b = a + 1$ and $c = b + 1$

$\Rightarrow$ $c = a + 2 \Rightarrow (a, c) \notin R$

$\therefore$ R is not transitive.

Hence R is not reflexive, not symmetric and not transitive.

4. (i) **Reflexive :**

Let $a \in R, a \leq a$ which is true. $\therefore$ $(a, a) \in R$.

Thus R is reflexive.

(ii) **Symmetric :**

Let $a, b \in R$ and $(a, b) \in R$

Consider $a \leq b$ does not imply $b \leq a$

$\Rightarrow$ $(a, b) \in R$ but $(b, a) \notin R$

$\therefore$ R is not symmetric.

(iii) **Transitive :**

Let $a, b, c \in R$

If $(a, b) \in R$ and $(b, c) \in R \Rightarrow a \leq b$ and $b \leq c$

$\Rightarrow$ $a \leq c \Rightarrow (a, c) \in R$

Thus R is transitive.

Hence R is reflexive and transitive but not symmetric.

5. (i) **Reflexive :**

Let $a \in R, a \leq a^3$, which is false.

$\therefore$ $(a, a) \notin R$.
Thus, R is not reflexive.
(ii) **Symmetric :**
Let $a, b \in R$, and $(a, b) \in R$
Consider, $a \le b^3 \not\Rightarrow b \le a^3$ $\therefore$ $(b, a) \notin R$.
Thus, R is not symmetric.
(iii) **Transitive :**
Let $a, b, c \in R$, consider $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow$ $a \le b^3$ and $b \le c^3$
$\Rightarrow$ $a \le c^9 \Rightarrow (a, c) \notin R$
$\therefore$ R is not transitive.
Hence R is none of reflexive, symmetric and transitive.

6. Given the set {1, 2, 3} and
$R = \{(1, 2), (2, 1)\}$
(i) **Reflexive :**
As $1, 2, 3 \in \{1, 2, 3\}$ but $(1, 1) \notin R$, $(2, 2) \notin R$, $(3, 3) \notin R$
$\therefore$ R is not reflexive.
(ii) **Symmetric :**
For $1, 2 \in \{1, 2, 3\}$, $(1, 2) \in R$, we have $(2, 1) \in R$
$\therefore$ R is symmetric.
(iii) **Transitive :**
For $1, 2, 3 \in \{1, 2, 3\}$, $(1, 2) \in R$, $(2, 1) \in R$ but $(1, 1) \notin R$
$\therefore$ R is not transitive.
Hence R is symmetric but neither reflexive nor transitive.

7. $R = \{(x, y) : x$ and y have same number of pages$\}$
(i) **Reflexive :**
Books x and x have same number of pages.
$\therefore$ $(x, x) \in R$.
$\therefore$ R is reflexive.
(ii) **Symmetric :**
If $(x, y) \in R$, *i.e.*, Books x and y have same number of pages.
$\Rightarrow$ Books y and x have same number of pages
$\Rightarrow$ $(y, x) \in R$
$\therefore$ R is symmetric.
(iii) **Transitive :**
If $(x, y) \in R$ and $(y, z) \in R$
$\Rightarrow$ Books x and y have same number of pages and books y and z have same number of pages.
$\Rightarrow$ Books x and z have same number of pages.
$\Rightarrow$ $(x, z) \in R$
$\therefore$ R is transitive.
Hence R is an equivalence relation.

8. We have $A = \{1, 2, 3, 4, 5\}$ and
$R = \{(a, b) : |a - b|$ is even$\}$
(i) **Reflexive :**
For any $a \in A$, we have $|a - a| = 0$, which is even.
$\therefore$ $(a, a) \in R \ \forall \ a \in A$
So, R is reflexive.
(ii) **Symmetric :**
Let $a, b \in A$, $(a, b) \in R$, then $|a - b|$ is even
$\Rightarrow$ $|b - a|$ is even $\Rightarrow$ $(b, a) \in R$
Thus $(a, b) \in R \Rightarrow (b, a) \in R$
So, R is symmetric.
(iii) **Transitive :**
Let $a, b, c \in A$. Let $(a, b) \in R$ and $(b, c) \in R$.
$\Rightarrow$ $|a - b|$ is even and $|b - c|$ is even.
$\Rightarrow$ (a and b both are even or both are odd) and (b and c both are even or both are odd)
Case I : When b is even.
Let $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow$ $|a - b|$ is even and $|b - c|$ is even
$\Rightarrow$ a is even and c is even. [$\because$ b is even]
$\Rightarrow$ $|a - c|$ is even $\Rightarrow$ $(a, c) \in R$
Case II : When b is odd.
Let $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow$ $|a - b|$ is even and $|b - c|$ is even.
$\Rightarrow$ a is odd and c is odd [$\because$ b is odd]
$\Rightarrow$ $|a - c|$ is even $\Rightarrow$ $(a, c) \in R$
Thus $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$
So, R is transitive.
Hence, R is an equivalence relation.
We know that the difference of any two odd (even) natural numbers is always an even natural number.
$\therefore$ All the elements of set {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other.
We know that the difference of an even natural number and an odd natural number is an odd number.
$\therefore$ No element of {1, 3, 5} is related to any element of {2, 4}.

9. (i) $A = \{x \in Z : 0 \le x \le 12\}$
$\therefore A = \{0, 1, 2, 3, ..., 12\}$
We have $R = \{(a, b) : |a - b|$ is a multiple of 4$\}$
(a) **Reflexive :**
For any $a \in A \Rightarrow |a - a| = 0$ is a multiple of 4
Thus $(a, a) \in R$ and R is reflexive.
(b) **Symmetric :**
Any $a, b \in A$. Let $(a, b) \in R$
$\Rightarrow$ $|a - b|$ is a multiple of 4.
$\Rightarrow$ $|b - a|$ is a multiple of 4.
$\Rightarrow$ $(b, a) \in R$
Thus $(a, b) \in R \Rightarrow (b, a) \in R$
i.e., R is symmetric.
(c) **Transitive :**
Any $a, b, c \in A$
Let $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow$ $|a - b|$ is a multiple of 4 and $|b - c|$ is a multiple of 4.
Let $|a - b| = 4k_1$ and $|b - c| = 4k_2$
Now, $|a - c| = |a - b + b - c| = |4k_1 + 4k_2| = 4|k_1 + k_2|$
[$\because$ $|a - b| = 4k_1$ and $|b - c| = 4k_2$ where $k_1, k_2 \in Z$]

$\Rightarrow$ $|a - c|$ is a multiple of 4 $\Rightarrow$ $(a, c) \in R$

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

(ii) $R = \{(a, b) : a = b\}$

$\Rightarrow$ $R = \{(0, 0), (1, 1),......,(12, 12)\}$ and
$A = \{0, 1, 2,.........,12\}$

(a) **Reflexive :**

For $a \in A$, we have $a = a \Rightarrow (a, a) \in R$

$\Rightarrow$ R is reflexive.

(b) **Symmetric :**

For $a, b \in A$

Let $(a, b) \in R \Rightarrow a = b \Rightarrow b = a \Rightarrow (b, a) \in R$

$\Rightarrow$ R is symmetric.

(c) **Transitive :**

For $a, b, c \in A$. Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a = b$ and $b = c \Rightarrow a = c \Rightarrow (a, c) \in R$

$\Rightarrow$ R is transitive.

Hence R is an equivalence relation.

Now set of all elements related to 1 in each case.

For (i), required set = $\{(5, 1), (1, 5), (9, 1), (1, 9)\}$
i.e., $\{1, 5, 9\}$

For (ii), required set = $\{(1, 1)\}$ *i.e.*, $\{1\}$

10. (i) The relation R "is perpendicular to" *i.e.*, $R = \{(l_1, l_2) : l_1 \perp l_2\}$

Now, l_1 is not perpendicular to l_1.

$\therefore$ R is not reflexive.

If $l_1 \perp l_2$ then $l_2 \perp l_1$.

$\therefore$ R is symmetric.

If $l_1 \perp l_2$ and $l_2 \perp l_3$, then l_1 is not perpendicular to l_3.

$\therefore$ R is not transitive.

Clearly R "is perpendicular to" is a symmetric but neither reflexive nor transitive.

(ii) Relation $R = \{(x, y) : x > y\}$

We know that $x > x$ is false.

$\therefore$ R is not reflexive.

If $x > y$ does not imply $y > x$.

$\therefore$ R is not symmetric.

If $x > y$, $y > z$ implies $x > z$.

$\therefore$ R is transitive.

Thus R is transitive but neither reflexive nor symmetric.

(iii) Relation R "Is friend of "

i.e., $R = \{(x, y) : x$ is a friend of $y\}$

Now, x is a friend of x. $\therefore$ R is reflexive.

If x is a friend of y, then y is a friend of x.

$\therefore$ R is symmetric.

If x is a friend of y and y is a friend of z, then x need not be a friend of z.

$\therefore$ R is not transitive.

Thus, R is reflexive and symmetric but not transitive.

(iv) R is relation "Is greater or equal to"

i.e., $R = \{(x, y) : x \geq y\}$

As $x \geq x$ is true $\therefore$ R is reflexive.

If $x \geq y$ does not imply $y \geq x$

$\therefore$ R is not symmetric.

If $x \geq y, y \geq z \Rightarrow x \geq z$

$\therefore$ R is transitive.

Hence, R is reflexive and transitive but not symmetric.

(v) $R = \{(1, 2), (2, 1), (1, 1), (2, 2)\}$ on set $\{1, 2, 3\}$

Hence R is symmetric, transitive but not reflexive.

11. $R = \{(P, Q)$: distance of the point P from the origin is the same as the distance of the point Q from the origin$\}$

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ and $O(0, 0)$.

$\therefore$ $OP = OQ \Rightarrow \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$

$\Rightarrow$ $x_1^2 + y_1^2 = x_2^2 + y_2^2$

(a) **Reflexive :**

Let $P \in A$, then

Distance of the point P from origin is same as the distance of the point P from origin.

i.e., $OP = OP$

$\Rightarrow$ $(P, P) \in R$

$\therefore$ R is reflexive.

(b) **Symmetric :**

Let $P, Q \in A$, If $(P, Q) \in R$

$\Rightarrow$ Distance of the point P from origin is same as the distance of the point Q from origin *i.e.*, $OP = OQ$

$\Rightarrow$ $OQ = OP \Rightarrow (Q, P) \in R$

$\therefore$ R is symmetric.

(c) **Transitive :**

Let $P, Q, S \in A$, $(P, Q) \in R$ and $(Q, S) \in R$

$\Rightarrow$ $OP = OQ$ and $OQ = OS$

$\Rightarrow$ $OP = OS$

$\Rightarrow$ $(P, S) \in R$

$\therefore$ R is transitive.

Hence R is an equivalence relation.

We have to find the set of points related to $P \neq (0, 0)$.

As $x_1^2 + y_1^2 = x_2^2 + y_2^2 = r^2 \Rightarrow x^2 + y^2 = r^2$

which represents a circle with centre $(0, 0)$ and radius = r.

12. $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$

(a) **Reflexive :**

We know that each triangle is similar to itself and thus $(T_1, T_1) \in R$.

$\therefore$ R is reflexive.

(b) **Symmetric :**

Let $(T_1, T_2) \in R$

Then $T_1 \sim T_2 \Rightarrow T_2 \sim T_1$.

$\therefore$ R is symmetric.

(c) **Transitive :**

Let $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

then $T_1 \sim T_2$ and $T_2 \sim T_3$

$\Rightarrow \quad T_1 \sim T_3$.

$\therefore \quad R$ is transitive.

Hence R is an equivalence relation.

Now, we are given three right angled triangles T_1, T_2 and T_3.

T_1 with sides 3, 4, 5 ; T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10

We know that two triangles are similar if corresponding sides are proportional.

We observe that T_1 and T_3 are similar because $\frac{3}{6}=\frac{4}{8}=\frac{5}{10}\left(=\frac{1}{2}\right)$

Hence triangle T_1 and T_3 are related.

13. $R = \{(P_1, P_2) : P_1$ and P_2 have the same number of sides$\}$.

(a) **Reflexive :**

Let $P_1 \in A$

P_1 and P_1 have same number of sides.

$\Rightarrow \quad (P_1, P_1) \in R$. Hence R is reflexive.

(b) **Symmetric :**

Let $P_1, P_2 \in A$. If $(P_1, P_2) \in R$

$\Rightarrow \quad P_1$ and P_2 have same number of sides.

$\Rightarrow \quad P_2$ and P_1 have same number of sides.

$\Rightarrow \quad (P_2, P_1) \in R$. Hence R is symmetric

(c) **Transitive :**

Let $P_1, P_2, P_3 \in A$. If $(P_1, P_2) \in R$ and $(P_2, P_3) \in R$

$\Rightarrow \quad P_1$ and P_2 have same number of sides

and P_2 and P_3 have same number of sides

$\Rightarrow \quad P_1$ and P_3 have same number of sides

$\Rightarrow \quad (P_1, P_3) \in R$

Thus R is transitive. Hence R is an equivalence relation.

We know that if 3, 4, 5 are the sides of a Δ, then the triangle is right-angled. Now the set A is the set of all triangles.

14. $R = \{(L_1, L_2) : L_1$ is parallel to $L_2\}$

(a) **Reflexive :**

Let $L_1 \in L$, $L_1 \,||\, L_1$ *i.e.*, $(L_1, L_1) \in R$. Thus R is reflexive.

(b) **Symmetric :**

Let $L_1, L_2 \in L$, $(L_1, L_2) \in R \Rightarrow L_1 \,||\, L_2$ then $L_2 \,||\, L_1$

$\Rightarrow \quad (L_2, L_1) \in R$

Thus R is symmetric.

(c) **Transitive :**

Let $L_1, L_2, L_3 \in L$, $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$

$\Rightarrow \quad L_1 \,||\, L_2$ and $L_2 \,||\, L_3 \Rightarrow L_1 \,||\, L_3$

Clearly, R is transitive. Hence, R is an equivalence relation.

Also, all lines related to the line $y = 2x + 4$ are $y = 2x + c$, where c is a real number.

15. **(B):** R is reflexive because (1, 1), (2, 2), (3, 3), (4, 4) $\in R$ for all 1, 2, 3, 4 $\in \{1, 2, 3, 4\}$

R is not symmetric because $(1, 2) \in R$ but $(2, 1) \notin R$ for all $1, 2 \in \{1, 2, 3, 4\}$

R is transitive because $\forall\, x, y, z \in \{1, 2, 3, 4\}$, we have

$(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow \quad (x, z) \in R$

16. **(C):** $(a, b) \in R$ if only if $a = b - 2$ and $b > 6$. $(6, 8) \in R$ as $6 = 8 - 2$ and $8 > 6$.

EXERCISE - 1.2

1. $f : R_* \to R_*$ defined by $f(x)=\frac{1}{x}$

(i) Let $f(x_1)=\frac{1}{x_1}$ and $f(x_2)=\frac{1}{x_2}$

If $f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1}=\frac{1}{x_2} \Rightarrow x_1 = x_2$

Thus f is one-one.

(ii) Since $f : R_* \to R_*$

Given any element $y \in R_*$ (co-domain of f), then there exist an element $x \in R_*$ (domain of f), such that

$f(x) = y$ and we have $f(x)=\frac{1}{x}$

$\Rightarrow \frac{1}{x}=y \Rightarrow x=\frac{1}{y} \Rightarrow f^{-1}(y)=\frac{1}{y}$

$\Rightarrow f\left(\frac{1}{y}\right)=y$

Thus f is onto. Hence f is a one-one and onto function.

Now, this result is not true, since if domain R_* is replaced by N with co-domain being same as R_* then f does not have its inverse.

2. (i) $f : N \to N$ given by $f(x) = x^2$

(a) $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$

(as x_1 and x_2 are natural numbers)

$\therefore \quad f$ is one-one *i.e.*, f is injective.

(b) There are some elements in co-domain which have no pre-image in domain N.

e.g., $3 \in$ co-domain N, but there is no pre-image of 3 in the domain of f.

Thus f is not onto *i.e.*, f is not surjective.

Hence f is injective but not surjective.

(ii) $f : Z \to Z$ given by $f(x) = x^2$,

where $Z = \{0, \pm1, \pm2, \pm3, ...\}$

(a) Let $-1, 1 \in Z$, $f(-1) = f(1) \Rightarrow (-1)^2 = (1)^2 = 1 = 1$

But $-1 \neq 1$. $\therefore$ f is not one-one *i.e.*, f is not injective.

(b) There are many such elements belongs to co-domain which have no pre-image in its domain Z.

e.g., $2 \in Z$ (co-domain). But $2^{1/2} \notin Z$ (domain)

$\therefore \quad$ Element 2 has no pre-image in its domain Z.

$\therefore \quad f$ is not onto *i.e.*, f is not surjective.

Hence f is neither injective nor surjective.

(iii) $f : R \to R$ given by $f(x) = x^2$

(a) Let –1, 1 ∈ R, $f(-1) = f(1) \Rightarrow 1 = 1$
But $-1 \neq 1$. ∴ f is not injective.
(b) –5 belongs to co-domain R of f, but $\sqrt{-5}$ does not belong to domain R of f.
Hence f is neither injective nor surjective.
(iv) $f: N \to N$ given by $f(x) = x^3$
(a) Let $x_1, x_2 \in N, f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3$
$\Rightarrow x_1 = x_2$
i.e., for every $x \in N$, has a unique image in its co-domain.
∴ f is one-one *i.e.*, f is injective.
(b) There are many such members of co-domain of f which do not have any pre-image in its domain. *e.g.*, 2, 3 etc.
Thus f is not onto *i.e.*, f is not surjective.
Hence f is injective but not surjective.
(v) $f: Z \to Z$, given by $f(x) = x^3$
(a) $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$
∴ f is one-one *i.e.*, f is injective.
(b) Many members of co-domain of f have no pre-image in its domain. *e.g.*, 2 belongs to its co-domain has no pre-image in its domain of f.
Therefore f is not surjective.
Hence f is injective but not surjective.

3. $f: R \to R, f(x) = [x]$,
∴ $f(1.2) = 1$ & $f(1.5) = 1 \Rightarrow f(1.2) = f(1.5)$
but $1.2 \neq 1.5$
∴ f is not one-one.
$f: R \to R$ does not attain non-integral values.
∴ Non-integer points in co-domain R do not have their pre-image in the domain R.
∴ f is not onto.
Hence f is neither one-one nor onto.

4. $f: R \to R$, given by $f(x) = |x|$

$$\text{Define } |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Clearly f contains (–1, 1), (1, 1), (–2, 2), (2, 2)......
We have $f(-1) = f(1) \Rightarrow |-1| = |1| \Rightarrow 1 = 1$
But $(-1) \neq (1)$
∴ f is not one-one.
We have $f: R \to R$, $f(x) = |x|$ assumes only non-negative values. So, negative real numbers in R (co-domain) do not have their pre-image in R (domain).
∴ f is not onto. Hence f is neither one-one nor onto.

5. $$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

(a) We have $f(1) = f(2) = 1$ (∵ $1, 2 > 0$)
But $1 \neq 2$
Also, $f(-2) = f(-3) = -1$. But $-2 \neq -3$
$\Rightarrow$ f is not one-one.
(b) Except the number –1, 0, 1, no other members of co-domain of f has any pre-image in its domain.
∴ f is not onto.
Hence f is neither one-one nor onto.

6. $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and
$f = \{(1, 4), (2, 5), (3, 6)\}$
We have $f(1) = 4$, $f(2) = 5$ and $f(3) = 6$ *i.e.*, distinct elements of A have distinct images in B. Hence f is one-one.

7. (i) $f: R \to R$ defined by $f(x) = 3 - 4x$
Now, $f(x_1) = f(x_2) \Rightarrow 3 - 4x_1 = 3 - 4x_2$
$\Rightarrow -4x_1 = -4x_2 \Rightarrow x_1 = x_2$
∴ f is one-one.
$f: R \to R$ be given for every $y \in R$(co-domain of f), there exists an element $x \in R$ (domain of f) such that

$$f(x) = y \Rightarrow y = 3 - 4x \Rightarrow x = \frac{3-y}{4}$$

$$\therefore\ f\left(\frac{3-y}{4}\right) = 3 - 4\left(\frac{3-y}{4}\right) = 3 - 3 + y = y$$

Hence f is onto. Thus f is one-one and onto and hence bijective.
(ii) $f: R \to R$ defined by $f(x) = 1 + x^2$.
Let $x_1, x_2 \in R$, then
$f(x_1) = 1 + x_1^2$ and $f(x_2) = 1 + x_2^2$
Consider, $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$
$\Rightarrow x_1 = \pm x_2$
Thus f is not one-one.
Now, $f: R \to R$, given for every $y \in R$(co-domain of f), there exists an element $x \in R$(domain of f) such that $f(x) = y$
$\Rightarrow y = 1 + x^2 \Rightarrow x = \pm\sqrt{y-1}$
$\Rightarrow y \geq 1 \Rightarrow$ Range $= [1, \infty) \neq R$
∴ f is not onto.
Hence f is neither one-one nor onto and hence not bijective.

8. **Injectivity :** Let (a_1, b_1) and $(a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$
$\Rightarrow (b_1, a_1) = (b_2, a_2) \Rightarrow b_1 = b_2$ and $a_1 = a_2$
$\Rightarrow (a_1, b_1) = (a_2, b_2)$
Thus, $f(a_1, b_1) = f(a_2, b_2) \Rightarrow (a_1, b_1) = (a_2, b_2)$
[for all $(a_1, b_1), (a_2, b_2) \in A \times B$.]
So, f is injective.
Surjectivity : Let (b, a) be an arbitrary element of $B \times A$, where $b \in B$ and $a \in A \Rightarrow (a, b) \in A \times B$
Thus, for all $(b, a) \in B \times A$, their exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$
So, $f: A \times B \to B \times A$ is an onto function.
Hence, f is bijective.

9. Here, $f(1) = \frac{1+1}{2} = 1,\ f(2) = \frac{2}{2} = 1$
$\Rightarrow f(1) = f(2)$ but $1 \neq 2$

$\Rightarrow$ f is not one-one.

But f is onto because range of $f = N$

[$\because$ For any $x \in \mathrm{N}, 2x \in \mathrm{N}$ such that $f(2x) = \frac{2x}{2} = x$]

$\Rightarrow$ f is onto. Hence f is not bijective.

10. $A = R - \{3\}$, $B = R - \{1\}$ and $f(x) = \frac{x-2}{x-3}$

Let $x_1, x_2 \in A$.

$\therefore$ $f(x_1) = \frac{x_1 - 2}{x_1 - 3}$ and $f(x_2) = \frac{x_2 - 2}{x_2 - 3}$

Consider, $f(x_1) = f(x_2)$

$\Rightarrow$ $\frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$

$\Rightarrow$ $(x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$

$\Rightarrow$ $x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow$ $-3x_1 - 2x_2 = -2x_1 - 3x_2$ $\Rightarrow$ $x_1 = x_2$

$\therefore$ f is one-one.

Now, $f : A \to B$, let $y \in B$ (co-domain of f) be any element, then there exist $x \in A$ (domain of f) such that $f(x) = y$

$\Rightarrow$ $y = \frac{x-2}{x-3}$

$\Rightarrow$ $y(x - 3) = x - 2 \Rightarrow xy - 3y = x - 2 \Rightarrow x(y - 1) = 3y - 2$

$\Rightarrow$ $x = \frac{3y-2}{y-1}$ $\quad \therefore f\left(\frac{3y-2}{y-1}\right) = \frac{\frac{3y-2}{y-1} - 2}{\frac{3y-2}{y-1} - 3}$

$= \frac{3y - 2 - 2y + 2}{3y - 2 - 3y + 3} = y$

$\therefore$ f is onto.

Hence f is one-one and onto.

11. **(D) :** $f : R \to R$ defined as $f(x) = x^4$

Let $x_1, x_2 \in R$.

Consider $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1^4 = x_2^4$ $\Rightarrow$ $x_1^2 = x_2^2$ $\Rightarrow \pm x_1 = \pm x_2$

$\therefore$ f is not one-one.

The negative elements in co-domain (R) do not have their pre-image in domain (R).

$\therefore$ f is not onto.

Thus f is neither one-one nor onto.

12. **(A) :** Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow$ $3x_1 = 3x_2$ $\Rightarrow$ $x_1 = x_2$

$\Rightarrow$ f is one-one

Consider any $y \in R$ (co-domain of f), there exist $x \in R$ (domain of f) such that

$f(x) = y \Rightarrow 3x = y \Rightarrow x = \frac{y}{3}$

$\therefore$ $f\left(\frac{y}{3}\right) = 3 \cdot \frac{y}{3} = y$

$\Rightarrow$ f is onto

$\therefore$ Hence, f is one-one onto

NCERT MISCELLANEOUS EXERCISE

1. We have : $f(x) = \frac{x}{1+|x|} = \begin{cases} \frac{x}{1+x}, & \text{if } x \ge 0 \\ \frac{x}{1-x}, & \text{if } x < 0 \end{cases}$

Here, domain of $f = R$ [$\because 1 + |x| \ne 0 \ \forall \ x \in R$]

To prove : f is one-one.

Let $x, y \in$ Domain of $f = R$, such that $x \ne y$

Here four cases arise.

Case I : When $x \ge 0, y \ge 0$

Then $x \ne y \Rightarrow 1 + x \ne 1 + y$

$\Rightarrow$ $\frac{1}{1+x} \ne \frac{1}{1+y}$ $\Rightarrow$ $\frac{-1}{1+x} \ne \frac{-1}{1+y}$

$\Rightarrow$ $1 - \frac{1}{1+x} \ne 1 - \frac{1}{1+y}$ $\Rightarrow$ $\frac{x}{1+x} \ne \frac{y}{1+y}$

$\Rightarrow$ $f(x) \ne f(y)$.

Case II : When $x \ge 0$ and $y < 0$

Then $f(x) = \frac{x}{1+x} \ge 0$ and $f(y) = \frac{y}{1-y} < 0$

$\Rightarrow$ $f(x) \ne f(y)$.

Case III : When $x < 0$ and $y \ge 0$

Then $f(x) < 0$ and $f(y) \ge 0$ [As in Case II]

$\Rightarrow$ $f(x) \ne f(y)$

Case IV : When $x \le 0$ and $y \le 0$

Then $x \ne y \Rightarrow -x \ne -y$

$\Rightarrow$ $1 - x \ne 1 - y$ $\Rightarrow$ $\frac{1}{1-x} \ne \frac{1}{1-y}$

$\Rightarrow$ $\frac{1}{1-x} - 1 \ne \frac{1}{1-y} - 1$ $\Rightarrow$ $\frac{x}{1-x} \ne \frac{y}{1-y}$

$\Rightarrow$ $f(x) \ne f(y)$.

Thus, in each case, $x \ne y \Rightarrow f(x) \ne f(y)$.

Hence f is one-one.

To prove : f is onto.

Let $y \in (-1, 1)$, where y is arbitrary.

Then $y = f(x) = \frac{x}{1+|x|} = \begin{cases} \frac{x}{1+x} < 1, & \text{if } x \ge 0 \\ \frac{x}{1-x} > -1, & \text{if } x < 0 \end{cases}$

(I) When $y = \frac{x}{1+x}$, where $y \ge 0$

$\Rightarrow$ $y + xy = x \Rightarrow y = x(1 - y) \Rightarrow x = \frac{y}{1-y} > 0$

(II) When $y = \frac{x}{1-x}$, where $y < 0$

$\Rightarrow y - xy = x \Rightarrow y = x + xy \Rightarrow x = \frac{y}{1+y} < 0$

Thus when $y \ge 0$, there is $\frac{y}{1-y} \in$ Domain of $f = R$ such that $f\left(\frac{y}{1-y}\right)=\frac{\frac{y}{1-y}}{1+\frac{y}{1-y}}=\frac{y}{1-y+y}=\frac{y}{1}=y$

and when $y<0$, there is $\frac{y}{1+y}$ $\in$ Domain of $f = R$ such

that $f\left(\frac{y}{1+y}\right)=\frac{\frac{y}{1+y}}{1-\frac{y}{1+y}}=\frac{y}{1+y-y}=\frac{y}{1}=y$

Hence, f is onto.

2. Let $x_1, x_2 \in R$ be such that

$f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one. Hence $f(x) = x^3$ is injective.

3. (i) Since $A \subset A\ \forall\ A \in P(X) \Rightarrow ARA$

$\therefore$ R is reflexive.

(ii) Let $ARB \Rightarrow A \subset B$ and $BRA \Rightarrow B \subset A$

$\Rightarrow$ $A = B$ (which is not so) $[\because A \subset B \not\Rightarrow B \subset A]$

$\Rightarrow$ $ARB \not\Rightarrow BRA \Rightarrow R$ is not symmetric

(iii) $ARB, BRC \Rightarrow A \subset B, B \subset C$

$\Rightarrow$ $A \subset C \Rightarrow ARC \Rightarrow R$ is transitive

$\therefore$ R is not an equivalence relation of $P(X)$.

4. The number of onto functions that can be defined from a finite set X containing n elements onto a finite set Y containing n elements.

Let $X : \{1, 2, ..., n\}$ and $Y : \{1, 2, 3, ..., n\}$

One of the elements (say 1) has any one of the pre image 1, 2, ..., n *i.e.* n ways.

On similar way the element (say 2) in $(n - 1)$ ways.

$\therefore$ Total number of possible ways $= n(n-1)(n-2)..... 3\cdot2\cdot1 = n!$

5. When $x = -1 : f(-1) = 1^2 + 1 = 2$ and

$g(-1)=2\left|-1-\frac{1}{2}\right|-1=2$

When $x = 0 : f(0) = 0$ and

$g(0)=2\left|-\frac{1}{2}\right|-1=2\times\frac{1}{2}-1=0$

When $x = 1 : f(1) = 1^2 - 1 = 0$ and

$g(1)=2\left|1-\frac{1}{2}\right|-1=2\times\frac{1}{2}-1=0$

When $x = 2 : f(2) = 2^2 - 2 = 2$ and

$g(2)=2\left|2-\frac{1}{2}\right|-1=3-1=2$

Thus for each $a \in A$, $f(a) = g(a)$.

Hence, f and g are equal functions.

6. **(A) :** There is only one relation containing (1, 2) and (1, 3) which is reflexive and symmetric but not transitive.

7. **(B) :** There are two equivalence relations containing (1, 2).