EXAM DRILL

Inverse Trigonometric Functions

CHAPTER 2

SOLUTIONS

1. **(a) :** Given, $f(x)=\sin^{-1}\sqrt{x-1}$

Since, $x-1\geq 0$ and $-1\leq\sqrt{x-1}\leq 1$

$\therefore\ 0\leq x-1\leq 1$

$\Rightarrow\ 1\leq x\leq 2$

2. **(b) :** We have, $2\sec^{-1}2+\sin^{-1}\frac{1}{2}$

$=2\cdot\frac{\pi}{3}+\frac{\pi}{6}$

$=\frac{5\pi}{6}$

3. **(a) :** Given, $\alpha=\tan^{-1}(1)=\frac{\pi}{4}$

$\Rightarrow\ \pi=4\alpha$ ...(i)

Also, $\beta=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$

$\Rightarrow\ \pi=3\beta$ ...(ii)

From (i) and (ii), $4\alpha=3\beta$

4. **(c) :** The domain of $\cos^{-1}x$ is $[-1, 1]$ and $\sin^{-1}x$ is $[-1, 1]$.

$\therefore\ \cos^{-1}2x$ is defined for all x satisfying $-1\leq 2x\leq 1$

$\Rightarrow\ \frac{-1}{2}\leq x\leq\frac{1}{2}$

Thus, domain of $2\cos^{-1}2x+\sin^{-1}x$ is $\left[\frac{-1}{2},\frac{1}{2}\right]$.

5. **(c) :** Given, $4\left\{\frac{\pi}{2}-\sin^{-1}x\right\}+\sin^{-1}x=\pi$

$\Rightarrow\ 2\pi-3\sin^{-1}x=\pi$

$\Rightarrow\ 3\sin^{-1}x=\pi\ \Rightarrow\ \sin^{-1}x=\frac{\pi}{3}\ \Rightarrow\ x=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

6. **(c) :** Consider, $\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$

Let $\sin^{-1}\frac{\sqrt{63}}{8}=x\ \Rightarrow\ \sin x=\frac{\sqrt{63}}{8}$

$\Rightarrow\ \cos x=\frac{1}{8}$

Now, $\cos\frac{x}{2}=\sqrt{\frac{1+\cos x}{2}}=\sqrt{\frac{1+1/8}{2}}=\frac{3}{4}$

$\therefore\ \sin\frac{x}{4}=\sqrt{\frac{1-\cos\frac{x}{2}}{2}}=\sqrt{\frac{1-\frac{3}{4}}{2}}=\sqrt{\frac{1}{8}}=\frac{1}{2\sqrt{2}}$

7. We have, $\tan^{-1}x=\frac{\pi}{10}$

$\Rightarrow\ x=\tan\frac{\pi}{10}=\cot\left(\frac{\pi}{2}-\frac{\pi}{10}\right)$

$\Rightarrow\ x=\cot\left(\frac{2\pi}{5}\right)\ \Rightarrow\ \cot^{-1}x=\frac{2\pi}{5}$

8. One branch of $\cos^{-1}$ other than the principal value branch corresponds to $[2\pi, 3\pi]$.

9. Let $\cot^{-1}(-x)=t$ ...(i)

$\Rightarrow\ x=-\cot t\Rightarrow x=\cot(\pi-t)\Rightarrow\cot^{-1}x=\pi-t$

$\Rightarrow\ \cot^{-1}x=\pi-\cot^{-1}(-x)$ (From (i))

$\Rightarrow\ \cot^{-1}(-x)=\pi-\cot^{-1}x$

10. Let $\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)=y\ \Rightarrow\ \sin y=\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$ ...(i)

Now, $\sin 15°=\sin(60°-45°)$

$=\sin 60°\cdot\cos 45°-\cos 60°\cdot\sin 45°$

$=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}-\frac{1}{2}\times\frac{1}{\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}=\sin y$ (Using (i))

$\Rightarrow\ y=15°\ \Rightarrow\ y=\frac{\pi}{12}$

11. Let $\angle CAB=\alpha$, then $\angle DAB=2\alpha$ and $\angle EAB=3\alpha$

(i) (b) : In right ΔCAB,

$\tan\alpha=\frac{BC}{AB}=\frac{10}{20}=\frac{1}{2}$

$\Rightarrow\ \angle CAB=\alpha=\tan^{-1}\left(\frac{1}{2}\right)$

(ii) (c) : $\angle DAB=2\alpha=2\tan^{-1}(1/2)$

(iii) (d) : $\angle EAB=3\alpha=3\tan^{-1}(1/2)$

(iv) (b) : In riwght $\Delta CA'B$

$\tan\angle A'=\frac{BC}{A'B}=\frac{10}{25}=\frac{2}{5}$

$\Rightarrow\ \tan\angle CA'B=\tan^{-1}\left(\frac{2}{5}\right)$

$\therefore$ Required difference $=\tan^{-1}\left(\frac{1}{2}\right)-\tan^{-1}\left(\frac{2}{5}\right)$

(v) (c) : Domain and range of $\tan^{-1}x$ are R and $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ respectively.

12. L.H.S. $= \cot^{-1}\left(\frac{xy+1}{x-y}\right) + \cot^{-1}\left(\frac{yz+1}{y-z}\right) + \cot^{-1}\left(\frac{zx+1}{z-x}\right)$

$= \tan^{-1}\left(\frac{x-y}{1+xy}\right) + \tan^{-1}\left(\frac{y-z}{1+yz}\right) + \tan^{-1}\left(\frac{z-x}{1+zx}\right)$ $\left[\because \cot^{-1}x = \tan^{-1}\frac{1}{x}\right]$

$= (\tan^{-1}x - \tan^{-1}y) + (\tan^{-1}y - \tan^{-1}z) + (\tan^{-1}z - \tan^{-1}x)$

= 0 = R.H.S.

OR

The given equation is

$\cos(2\sin^{-1}x) = \frac{1}{9} \quad (x > 0)$ ...(i)

Put $\sin^{-1}x = \theta \Rightarrow x = \sin\theta$

$\therefore$ From (i), $\cos 2\theta = \frac{1}{9} \Rightarrow 1 - 2\sin^2\theta = \frac{1}{9}$

$\Rightarrow 2\sin^2\theta = 1 - \frac{1}{9} = \frac{8}{9}$

$\Rightarrow x^2 = \frac{4}{9} \Rightarrow x = \frac{2}{3} \quad (\because x > 0)$

13. Consider, $\tan^{-1}\left\{2\sin\left(4\cos^{-1}\frac{\sqrt{3}}{2}\right)\right\}$

$= \tan^{-1}\left\{2\sin\left\{4\times\frac{\pi}{6}\right\}\right\} = \tan^{-1}\left\{2\sin\left(\frac{2\pi}{3}\right)\right\}$

$= \tan^{-1}\left\{2\times\frac{\sqrt{3}}{2}\right\} = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

14. Let $\cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\} = y \Rightarrow \cos y = \frac{x}{\sqrt{x^2+a^2}}$

Putting $x = a\cot\theta$, we get

$\cos y = \frac{a\cot\theta}{\sqrt{a^2\cot^2\theta + a^2}}$

$\Rightarrow \cos y = \frac{a\cot\theta}{a\operatorname{cosec}\theta}$

$\Rightarrow \cos y = \cos\theta \Rightarrow y = \theta$

$\Rightarrow \cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\} = \cot^{-1}\left(\frac{x}{a}\right)$

$\left\{\because x = a\cot\theta \Rightarrow \cot\theta = \frac{x}{a} \Rightarrow \cot^{-1}\frac{x}{a} = \theta\right\}$

15. Let, $\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\} = y, 0 < x < 1$ ...(i)

Putting $x = \cos\theta$ in (i), we get

$\sin y = \frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{2}$

$\Rightarrow \sin y = \frac{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}{2}$

$\Rightarrow \sin y = \sqrt{2}\,\frac{\left\{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right\}}{2}$

$\Rightarrow \sin y = \left\{\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}+\frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right\} \Rightarrow \sin y = \sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)$

$\Rightarrow y = \frac{\theta}{2}+\frac{\pi}{4} = \frac{1}{2}\cos^{-1}x + \frac{\pi}{4}$

16. Since, we know that $\sin^{-1}x \le \frac{\pi}{2}, x \in [-1, 1]$

$\Rightarrow (\sin^{-1}x)^2 \le \frac{\pi^2}{4}, x \in [-1, 1]$ ...(i)

Similarly, $(\sin^{-1}y)^2 \le \frac{\pi^2}{4}, y \in [-1, 1]$ ...(ii)

and $(\sin^{-1}z)^2 \le \frac{\pi^2}{4}, z \in [-1, 1]$ ...(iii)

Now, given $\frac{3\pi^2}{4} = (\sin^{-1}x)^2 + (\sin^{-1}y)^2 + (\sin^{-1}z)^2$

$\le \frac{3\pi^2}{4}$ [From (i), (ii) and (iii)]

$\Rightarrow \sin^{-1}x = \pm\frac{\pi}{2}, \sin^{-1}y = \pm\frac{\pi}{2}, \sin^{-1}z = \pm\frac{\pi}{2}$

$\Rightarrow x = \pm 1, y = \pm 1, z = \pm 1$

$\therefore x^2 + y^2 + z^2 = 1^2 + 1^2 + 1^2 = 3$

17. Consider,

L.H.S. $= \tan^{-1}\left(\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}\cdot\frac{1}{c_1}}\right) + \tan^{-1}\left(\frac{\frac{1}{c_1}-\frac{1}{c_2}}{1+\frac{1}{c_1c_2}}\right) + \ldots + \tan^{-1}\left(\frac{\frac{1}{c_{n-1}}-\frac{1}{c_n}}{1+\frac{1}{c_{n-1}c_n}}\right) + \tan^{-1}\frac{1}{c_n}$

$= \left(\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{1}{c_1}\right) + \left(\tan^{-1}\frac{1}{c_1} - \tan^{-1}\frac{1}{c_2}\right) + \left(\tan^{-1}\frac{1}{c_2} - \tan^{-1}\frac{1}{c_3}\right) + \ldots + \left(\tan^{-1}\frac{1}{c_n-1} - \tan^{-1}\frac{1}{c_n}\right) + \tan^{-1}\frac{1}{c_n}$

$= \tan^{-1}\frac{x}{y}$ = R.H.S.

OR

Given, $\cos^{-1}\left(\frac{xy}{6} - \sqrt{1-\frac{x^2}{9}}\sqrt{1-\frac{y^2}{4}}\right) = \frac{\theta}{2}$

$$\Rightarrow \quad \frac{xy}{6} - \sqrt{\frac{9-x^2}{9}}\sqrt{\frac{4-y^2}{4}} = \cos\frac{\theta}{2}$$

$$\Rightarrow \quad \frac{xy}{6} - \cos\frac{\theta}{2} = \frac{\sqrt{9-x^2}\sqrt{4-y^2}}{6}$$

$$\Rightarrow \quad xy - 6\cos\frac{\theta}{2} = \sqrt{9-x^2}\sqrt{4-y^2}$$

Squaring both sides, we have

$$x^2y^2 + 36\cos^2\frac{\theta}{2} - 12xy\cos\frac{\theta}{2} = 36 - 9y^2 - 4x^2 + x^2y^2$$

$$\Rightarrow \quad 4x^2 + 9y^2 - 12xy\cos\frac{\theta}{2} = 36\left(1-\cos^2\frac{\theta}{2}\right)$$

$$\Rightarrow \quad 4x^2 + 9y^2 - 12xy\cos\frac{\theta}{2} = 36\left(1-\left(\frac{\cos\theta+1}{2}\right)\right)$$

$$\Rightarrow \quad 4x^2 + 9y^2 - 12xy\cos\frac{\theta}{2} = 18(1-\cos\theta)$$

18. We have,

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$$

Let $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = x \Rightarrow \tan x = \frac{-1}{\sqrt{3}} = -\tan\frac{\pi}{6}$

$$\Rightarrow \quad \tan x = \tan\left(\pi - \frac{\pi}{6}\right) = \tan\frac{5\pi}{6}$$

$$\Rightarrow \quad x = \frac{5\pi}{6}$$

Also, $\cot^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{3}$

Similarly, $\tan^{-1}\left[\sin\left(\frac{\pi}{2}\right)\right] = \tan^{-1}(1) = \pi/4$

$$\therefore \quad \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\sin\frac{\pi}{2}\right)$$

$$= \frac{5\pi}{6} + \frac{\pi}{3} + \frac{\pi}{4} = \frac{17\pi}{12}$$

19. We have,

$$\tan^{-1}\left(\frac{1}{1+1\cdot 2}\right) + \tan^{-1}\left(\frac{1}{1+2\cdot 3}\right) + \ldots + \tan^{-1}\left(\frac{1}{1+n\cdot(n+1)}\right) = \tan^{-1}\theta$$

$$\Rightarrow \tan^{-1}\left(\frac{2-1}{1+1\cdot 2}\right) + \tan^{-1}\left(\frac{3-2}{1+2\cdot 3}\right) + \ldots + \tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right) = \tan^{-1}\theta$$

$$\Rightarrow \quad \tan^{-1}2 - \tan^{-1}1 + \tan^{-1}3 - \tan^{-1}2 + \ldots + \tan^{-1}(n+1) - \tan^{-1}(n) = \tan^{-1}\theta$$

$$\Rightarrow \quad \tan^{-1}(n+1) - \tan^{-1}1 = \tan^{-1}\theta$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{(n+1)-1}{1+(n+1)(1)}\right) = \tan^{-1}\theta$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{n}{n+2}\right) = \tan^{-1}\theta \Rightarrow \frac{n}{n+2} = \theta$$

OR

Consider,

$$\text{L.H.S.} = \tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\}$$

Let $\cos^{-1}\frac{a}{b} = \theta \Rightarrow \frac{a}{b} = \cos\theta$

$$\therefore \quad \text{L.H.S.} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$= \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} + \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$= \frac{1+\tan^2\frac{\theta}{2} + 1 + \tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = 2\left(\frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right)$$

$$= \frac{2}{\cos\left(2\cdot\frac{\theta}{2}\right)} = \frac{2}{\cos\theta} = \frac{2}{a/b} = \frac{2b}{a} = \text{R.H.S.}$$

20. We have,

$$\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad \ldots(1)$$

Let $\sin^{-1}\left(\frac{2a}{1+a^2}\right) = p \Rightarrow \sin p = \frac{2a}{1+a^2}$ ...(2)

Put $a = \tan\theta$ in (2), we get

$$\sin p = \frac{2\tan\theta}{1+\tan^2\theta} \Rightarrow \sin p = \sin 2\theta$$

$$\Rightarrow \quad p = 2\theta \Rightarrow p = 2\tan^{-1}a$$

So, $\sin^{-1}\left(\frac{2a}{1+a^2}\right) = 2\tan^{-1}a$ ...(3)

Similarly, $\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = q \Rightarrow \frac{1-a^2}{1+a^2} = \cos q$ ...(4)

$$\Rightarrow \quad \cos q = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta \quad (\text{Put } a = \tan\theta \text{ in (4)})$$

$$\Rightarrow \quad q = 2\theta$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = 2\tan^{-1}a \quad \ldots(5)$$

Now, let $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = m \Rightarrow \tan m = \frac{2x}{1-x^2}$ ...(6)

Put $x = \tan\phi$ in (6), we get

$\tan m = \dfrac{2\tan\phi}{1-\tan^2\phi} = \tan 2\phi$

$\Rightarrow \quad m = 2\phi$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = 2\tan^{-1}x$ ...(7)

From (3), (5) & (7), we get

$2\tan^{-1}a + 2\tan^{-1}a = 2\tan^{-1}x$

$\Rightarrow \quad 4\tan^{-1}a = 2\tan^{-1}x$

$\Rightarrow \quad 2\tan^{-1}a = \tan^{-1}x$ ...(8)

Put $\tan^{-1}a = t$ in (8), we get

$2t = \tan^{-1}x \Rightarrow \tan(2t) = x$

$\Rightarrow \quad x = \dfrac{2\tan t}{1-\tan^2 t} \Rightarrow x = \dfrac{2a}{1-a^2}$

21. Consider, R.H.S. $= 2\tan^{-1}\left(\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}\right)$

Let $\tan^{-1}x = \theta \Rightarrow \tan\theta = x$ ...(i)

Now, $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-x^2}{1+x^2}$ (Using (i))

$\Rightarrow \quad 2\tan^{-1}x = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$\therefore \quad$ R.H.S. $= \cos^{-1}\left\{\dfrac{1-\tan^2(\alpha/2)\tan^2(\beta/2)}{1+\tan^2(\alpha/2)\tan^2(\beta/2)}\right\}$

$= \cos^{-1}\left\{\dfrac{\cos^2(\alpha/2)\cos^2(\beta/2)-\sin^2(\alpha/2)\sin^2(\beta/2)}{\cos^2(\alpha/2)\cos^2(\beta/2)+\sin^2(\alpha/2)\sin^2(\beta/2)}\right\}$

$= \cos^{-1}\left\{\dfrac{(2\cos^2(\alpha/2))(2\cos^2(\beta/2))-(2\sin^2(\alpha/2))\cdot(2\sin^2(\beta/2))}{(2\cos^2(\alpha/2))(2\cos^2(\beta/2))+(2\sin^2(\alpha/2))\cdot(2\sin^2(\beta/2))}\right\}$

$= \cos^{-1}\left\{\dfrac{(1+\cos\alpha)(1+\cos\beta)-(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)+(1-\cos\alpha)(1-\cos\beta)}\right\}$

$= \cos^{-1}\left(\dfrac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}\right) =$ L.H.S.