CHAPTER 4

Determinants

NCERT FOCUS

SOLUTIONS

EXERCISE - 4.1

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2\times(-1)-(-5)\times(4) = -2+20 = 18$

2. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

$= \cos\theta \times \cos\theta - (\sin\theta) \times (-\sin\theta) = \cos^2\theta + \sin^2\theta = 1$

(ii) $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix} = (x^2-x+1)(x+1)-(x+1)(x-1)$

$= x^3+1-(x^2-1) = x^3+1-x^2+1 = x^3-x^2+2$

3. $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

$\text{L.H.S.} = |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 8-32 = -24$

$\text{R.H.S.} = 4|A| = 4\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 4(2-8) = -24$

Hence, $|2A| = 4|A|$

4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow 3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$

$\text{L.H.S.} = |3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix} = 3\begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix} - 0\begin{vmatrix} 0 & 6 \\ 0 & 12 \end{vmatrix} + 3\begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix}$

$= (3\times 36) - 0 + 3(0) = 108$

$\text{R.H.S.} = 27|A| = 27\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$

$= 27\left[1\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} - 0\begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\right] = 27[1(4)-0+0] = 108$

Hence, $|3A| = 27|A|$

5. (i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 3\begin{vmatrix} 0 & -1 \\ -5 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 3 & -5 \end{vmatrix}$

$= 3(0-5) + 1(0+3) - (2\times 0) = -15+3 = -12$

(ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$

$= 3(1+6) + 4(1+4) + 5(3-2) = 21+20+5 = 46$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix}$

$= 0-1(0-6)+2(-3) = 6-6 = 0$

(iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix}$

$= 2(0-5)+1(0+3)-2(0-6) = -10+3+12 = 5$

6. $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$

$= 1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$

$= 1(-9+12)-1(-18+15)-2(8-5) = 3+3-6 = 0$

7. (i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$\Rightarrow 2-20 = 2x^2-24 \Rightarrow -18 = 2x^2-24$

$\Rightarrow 2x^2 = 6 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$\Rightarrow 10-12 = 5x-6x \Rightarrow -2 = -x \Rightarrow x = 2$

8. **(B) :** $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

$\Rightarrow x^2-36 = 36-36 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

EXERCISE - 4.3

1. (i) Area of triangle $= \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$

$= \frac{1}{2}\left[1\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} - 0 + 1\begin{vmatrix} 6 & 0 \\ 4 & 3 \end{vmatrix}\right]$

$= \frac{1}{2}[1(0-3)+1(18-0)] = \frac{15}{2} = 7.5$ sq. units.

(ii) Area of triangle $= \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$

$$=\frac{1}{2}\left[2\begin{vmatrix}1 & 1\\8 & 1\end{vmatrix}-7\begin{vmatrix}1 & 1\\10 & 1\end{vmatrix}+1\begin{vmatrix}1 & 1\\10 & 8\end{vmatrix}\right]$$

$$=\frac{1}{2}[2(1-8)-7(1-10)+1(8-10)]$$

$$=\frac{1}{2}[-14+63-2]=\frac{47}{2}=23.5 \text{ sq. units}$$

(iii) Area of triangle $=\frac{1}{2}\begin{vmatrix}-2 & -3 & 1\\3 & 2 & 1\\-1 & -8 & 1\end{vmatrix}$

$$=\frac{1}{2}\left[-2\begin{vmatrix}2 & 1\\-8 & 1\end{vmatrix}+3\begin{vmatrix}3 & 1\\-1 & 1\end{vmatrix}+1\begin{vmatrix}3 & 2\\-1 & -8\end{vmatrix}\right]$$

$$=\frac{1}{2}[-20+12-22]=\frac{-30}{2}=-15$$

$\therefore$ Area = 15 square units. (As area > 0)

2. Area of $\Delta ABC=\begin{vmatrix}a & b+c & 1\\b & c+a & 1\\c & a+b & 1\end{vmatrix}$

$= a(c+a-a-b)-(b+c)(b-c)+1(ab+b^2-c^2-ac)$

$= ac-ab-b^2+c^2+ab+b^2-c^2-ac$

$= 0$

$\therefore$ Points A, B, C are collinear.

3. (i) Area of Δ = 4 sq. units

Again, area of $\Delta=\frac{1}{2}\begin{vmatrix}k & 0 & 1\\4 & 0 & 1\\0 & 2 & 1\end{vmatrix}=\frac{1}{2}[k(0-2)-0+1(8-0)]$

$$=\frac{1}{2}[-2k+8]=-k+4$$

Now, $-k+4=\pm 4 \Rightarrow -k+4=4$ or $-k+4=-4$

$\Rightarrow k=0$ or $k=8 \Rightarrow k=0, 8$

(ii) Given, area of Δ = 4 sq. units

Also, area of $\Delta=\frac{1}{2}\begin{vmatrix}-2 & 0 & 1\\0 & 4 & 1\\0 & k & 1\end{vmatrix}$

$$=\frac{1}{2}[-2(4-k)-0+1(0)]=-4+k$$

Now, $-4+k=\pm 4$

$\Rightarrow -4+k=4$ or $-4+k=-4 \Rightarrow k=0, 8$

4. (i) Equation of the line passing through (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix}x & y & 1\\x_1 & y_1 & 1\\x_2 & y_2 & 1\end{vmatrix}=0 \Rightarrow \begin{vmatrix}x & y & 1\\1 & 2 & 1\\3 & 6 & 1\end{vmatrix}=0$$

$\Rightarrow x(2-6)-y(1-3)+1(6-6)=0$

$\Rightarrow -4x+2y=0 \Rightarrow 2x-y=0$

Hence, $y=2x$ is the required line.

(ii) Equation of the line passing through (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix}x & y & 1\\x_1 & y_1 & 1\\x_2 & y_2 & 1\end{vmatrix}=0 \Rightarrow \begin{vmatrix}x & y & 1\\3 & 1 & 1\\9 & 3 & 1\end{vmatrix}=0$$

$\Rightarrow x(1-3)-y(3-9)+1(9-9)=0$

$\Rightarrow -2x+6y=0 \Rightarrow x-3y=0$

Hence, $x-3y=0$ is the required line.

5. **(D) :** Area of $\Delta=\frac{1}{2}\begin{vmatrix}2 & -6 & 1\\5 & 4 & 1\\k & 4 & 1\end{vmatrix}=\pm 35$

$\Rightarrow \frac{1}{2}[2(4-4)+(5-k)+1(20-4k)]=\pm 35$

$\Rightarrow \frac{1}{2}[30-6k+20-4k]=\pm 35$

$\Rightarrow \frac{1}{2}[50-10k]=\pm 35 \Rightarrow 25-5k=\pm 35$

$\therefore 25-5k=35$ or $25-5k=-35$

$\Rightarrow -5k=10$ or $5k=60 \Rightarrow k=-2$ or $k=12$

EXERCISE - 4.4

1. (i) Let $A=\begin{vmatrix}2 & -4\\0 & 3\end{vmatrix}$

Minor of the element a_{ij} is M_{ij}.

$M_{11}=3, M_{12}=0, M_{21}=-4, M_{22}=2$

For cofactors, we know that $C_{ij}=(-1)^{i+j}M_{ij}$

$\therefore C_{11}=3,\ C_{12}=0,\ C_{21}=4,\ C_{22}=2$

(ii) Let $A=\begin{vmatrix}a & c\\b & d\end{vmatrix}$

Minors : $M_{11}=d, M_{12}=b, M_{21}=c, M_{22}=a$

For cofactors, we know that $C_{ij}=(-1)^{i+j}M_{ij}$

$\therefore C_{11}=d, C_{12}=-b, C_{21}=-c, C_{22}=a$

2. (i) Let $A=\begin{vmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{vmatrix}$

$$M_{11}=\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}=1, M_{12}=\begin{vmatrix}0 & 0\\0 & 1\end{vmatrix}=0, M_{13}=\begin{vmatrix}0 & 1\\0 & 0\end{vmatrix}=0$$

$$M_{21}=\begin{vmatrix}0 & 0\\0 & 1\end{vmatrix}=0, M_{22}=\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}=1, M_{23}=\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}=0$$

$$M_{31}=\begin{vmatrix}0 & 0\\1 & 0\end{vmatrix}=0, M_{32}=\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}=0, M_{33}=\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}=1$$

For cofactors, we know that $C_{ij}=(-1)^{i+j}M_{ij}$

$C_{11}=1, C_{12}=0, C_{13}=0, C_{21}=0, C_{22}=1, C_{23}=0, C_{31}=0, C_{32}=0, C_{33}=1$

(ii) Let $A = \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

$M_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 + 1 = 11, M_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6,$

$M_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3,$

$M_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = -4, M_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2, M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$M_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = -20, M_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13$

$M_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5$

For cofactors, we know that $C_{ij} = (-1)^{i+j} M_{ij}$

$C_{11} = 11, C_{12} = -6, C_{13} = 3, C_{21} = 4, C_{22} = 2, C_{23} = -1,$
$C_{31} = -20, C_{32} = 13, C_{33} = 5$

3. $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$C_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = -(9 - 16) = 7,$

$C_{22} = (-1)^{2+2} \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7,$

$C_{23} = (-1)^{2+3} \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = -(10 - 3) = -7$

Now, $\Delta = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

$= 2 \times 7 + 0 \times 7 + 1 \times (-7) = 14 + 0 - 7 = 7.$

4. Let $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y,$

$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = -(z - x) = x - z,$

$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$

Now, $\Delta = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

$\Delta = zy(z - y) + zx(x - z) + xy(y - x)$

$= yz^2 - y^2z + zx^2 - z^2x + xy^2 - x^2y$

$= x^2z - x^2y + xy^2 - xz^2 + yz^2 - y^2z$

$= x^2(z - y) + x(y^2 - z^2) + yz(z - y)$

$= (z - y)[x^2 - x(y + z) + yz]$

$= (z - y)\{x(x - y) - z(x - y)\}$

$= (z - y)(x - y)(x - z) = (x - y)(y - z)(z - x)$

5. **(D) :** $\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

EXERCISE - 4.5

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Let C_{ij} be cofactors of a_{ij} in A. Then, the cofactors of elements of A are given by

$C_{11} = (-1)^{1+1}(4) = 4, C_{12} = (-1)^{1+2}(3) = -3,$

$C_{21} = (-1)^{2+1}(2) = -2, C_{22} = (-1)^{2+2}(1) = 1$

$\therefore \quad \text{adj } A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}' = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Let C_{ij} be cofactors of a_{ij} in A.

$C_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3,$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2 + 10) = -12,$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6,$

$C_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1 - 0) = 1,$

$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 + 4 = 5,$

$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0 - 2) = 2,$

$C_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11,$

$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5 - 4) = -1,$

$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2 = 5$

$\therefore \quad \text{adj } A = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}' = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

$\Rightarrow \quad |A| = -12 + 12 = 0$

Let C_{ij} be co-factors of a_{ij} in A. Then, the co-factors of elements of A are given by

$C_{11} = (-1)^{1+1}(-6) = -6$, $C_{12} = (-1)^{1+2}(-4) = 4$,

$C_{21} = (-1)^{2+1}(3) = -3$, $C_{22} = (-1)^{2+2}(2) = 2$

$$\text{adj } A = \begin{bmatrix} -6 & 4 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

$$\therefore \quad A \cdot \text{adj } A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(\text{adj } A) \cdot A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| I$

4. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Let C_{ij} be cofactors of a_{ij} in A. Then, the cofactors of elements of A are given by

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = 0,$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -(9+2) = -11,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 0,$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -(-3-0) = 3,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3-2 = 1,$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -(0+1) = -1,$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = 2,$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -(-2-6) = 8,$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 0+3 = 3$$

$$\therefore \text{ adj } A = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}' = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\text{Now, } A \cdot (\text{adj } A) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 3-1-2 & 2-8+6 \\ 0 & 9+0+2 & 6+0-6 \\ 0 & 3+0-3 & 2+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 11I$$

$$(\text{adj } A) \cdot A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 11I$$

$$\det A = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix} = 1(0) + 1(9+2) + 2(0) = 11$$

Hence, $A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = 11I = |A|\, I$.

5. Let $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

Then, $|A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6+8 = 14 \neq 0.$

So, A is a non-singular matrix and therefore it is invertible. Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$C_{11} = (-1)^{1+1}(3) = 3$, $C_{12} = (-1)^{1+2}(4) = -4$,

$C_{21} = (-1)^{2+1}(-2) = 2$, $C_{22} = (-1)^{2+2}(2) = 2$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

6. Let $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

Then, $|A| = \begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix} = -2+15 = 13 \neq 0.$

So, A is a non-singular matrix and therefore it is invertible. Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$C_{11} = (-1)^{1+1}(2) = 2$, $C_{12} = (-1)^{1+2}(-3) = 3$,

$C_{21} = (-1)^{2+1}(5) = -5$, $C_{22} = (-1)^{2+2}(-1) = -1$

$$\therefore \text{ adj } A = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}' = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

7. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Then, $|A| = 10 \neq 0$

Now, for cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = 10,\ C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = 0,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0,\ C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -10,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = 5,\ C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0,$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 2,\ C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -4,$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}' = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

8. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

Then, $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -3 - 0 = -3 \neq 0$

So, A is non-singular matrix and therefore, it is invertible. Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -3 - 0 = -3,$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 0 \\ 5 & -1 \end{vmatrix} = -(-3-0) = 3,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} = 6 - 15 = -9,$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} = 0,\ C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} = -1,$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -2,\ C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} = 0,$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0,\ C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = 3$$

$$\text{adj } A = \begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = -\frac{1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

9. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{vmatrix} = 2(-1-0) - 1(4-0) + 3(8-7)$

$= -2 - 4 + 3 = -3 \neq 0.$

So, A is a non-singular matrix and therefore, it is invertible. Let C_{ij} be cofactor of a_{ij} in A. Then the cofactors of elements of A are given by

$$C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1,\ C_{12} = (-1)^{1+2}\begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix} = -4,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 4 & -1 \\ -7 & 2 \end{vmatrix} = 8 - 7 = 1,$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -(1-6) = 5,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 3 \\ -7 & 1 \end{vmatrix} = 2 + 21 = 23,$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ -7 & 2 \end{vmatrix} = -(4+7) = -11,$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} = 3,\ C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = 12,$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = -6$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -1 & -4 & 1 \\ 5 & 23 & -11 \\ 3 & 12 & -6 \end{bmatrix}' = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{|A|}\text{adj } A = -\frac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

10. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

Then, $|A| = 1(8-6) + 1(0+9) + 2(0-6)$

$= 2 + 9 - 12 = -1 \neq 0$

$\therefore$ A is invertible.

Let C_{ij} be the cofactor of a_{ij} in A, then the cofactors of elements of A are given by

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = 8 - 6 = 2,$$

$C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -(0+9) = -9,$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = 0-6 = -6,$

$C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = -(-4+4) = 0,$

$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2,$

$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -(-2+3) = -1,$

$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = 3-4 = -1,$

$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -(-3-0) = 3,$

$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2+0 = 2$

$$A^{-1} = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-1}\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

11. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

$|A| = 1(-\cos^2\alpha - \sin^2\alpha) = -1 \neq 0.$

$\therefore A^{-1}$ exists.

$C_{ij} = (-1)^{i+j} M_{ij}$

$C_{11} = (-1)^{1+1}\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} = -\cos^2\alpha - \sin^2\alpha = -1,$

$C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} = 0,$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} = 0,$

$C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 0 \\ \sin\alpha & -\cos\alpha \end{vmatrix} = 0,$

$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & -\cos\alpha \end{vmatrix} = -\cos\alpha,$

$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = -\sin\alpha,$

$C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 0 \\ \cos\alpha & \sin\alpha \end{vmatrix} = 0,$

$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = -\sin\alpha,$

$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} = \cos\alpha$

$$\text{Now, adj } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

12. We have, $AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$

Since $|AB| = 67 \times 61 - 47 \times 87 = -2 \neq 0$

$\therefore \quad (AB)^{-1}$ exists and is given by

$$(AB)^{-1} = \frac{1}{|AB|}\text{adj}(AB) = -\frac{1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -61 & 87 \\ 47 & -67 \end{bmatrix}$$

Further $|A| = 15 - 14 = 1 \neq 0$ and $|B| = 54 - 56 = -2 \neq 0.$

Therefore A^{-1} and B^{-1} both exists and are given by

$$A^{-1} = 1\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}, B^{-1} = -\frac{1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

$$\therefore \quad B^{-1}A^{-1} = -\frac{1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -61 & 87 \\ 47 & -67 \end{bmatrix}$$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$.

13. $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

L.H.S. $= A^2 - 5A + 7I$

$$= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Hence proved.

Now, pre-multiplying by A^{-1} both sides, we get

$(A^{-1}A)A - 5A^{-1}A + 7A^{-1}I = A^{-1}O$

$\Rightarrow \quad IA - 5I + 7A^{-1} = O$

$\Rightarrow \quad A - 5I + 7A^{-1} = O \quad \Rightarrow \quad 7A^{-1} = 5I - A$

$$\Rightarrow \quad 7A^{-1} = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad 7A^{-1} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad 7A^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

14. We are given that $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

Now $A^2 + aA + bI = O$

$$\Rightarrow \quad \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$$

$$\Rightarrow \quad \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = O$$

$$\Rightarrow \quad \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a+b & 2a \\ a & a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 11+3a+b & 8+2a \\ 4+a & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad 4 + a = 0 \quad \Rightarrow \quad a = -4$

Also, $3 + a + b = 0 \;\Rightarrow\; b = -3 + 4 \;\Rightarrow\; b = 1$

15. We have, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$\Rightarrow \quad A^2 = AA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

Now, L.H.S. $= A^3 - 6A^2 + 5A + 11I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} + \begin{bmatrix} -24 & -12 & -6 \\ 18 & -48 & 84 \\ -42 & 18 & -84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence, $A^3 - 6A^2 + 5A + 11I = O$

Now, $A^3 - 6A^2 + 5A + 11I = O$

$\Rightarrow \; 11I = -A^3 + 6A^2 - 5A$ ... (i)

Pre multiplying (i) by A^{-1}, we get

$11A^{-1}I = -A^{-1}A^3 + 6A^{-1}A^2 - 5A^{-1}A$

$\Rightarrow 11A^{-1} = -A^2 + 6A - 5I$

$$\Rightarrow \quad A^{-1} = -\frac{1}{11}A^2 + \frac{6}{11}A - \frac{5}{11}I$$

$$= -\frac{1}{11}\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + \frac{6}{11}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - \frac{5}{11}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -4+6-5 & -2+6+0 & -1+6+0 \\ 3+6+0 & -8+12-5 & 14-18+0 \\ -7+12+0 & 3-6+0 & -14+18-5 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

16. We have $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$A^2 = AA = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now, $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+ 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9-0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence $A^3 - 6A^2 + 9A - 4I = O$

Now, $A^3 - 6A^2 + 9A - 4I = O$

$\Rightarrow \quad 4I = A^3 - 6A^2 + 9A$

Pre multiplying both sides by A^{-1}, we get

$4A^{-1}I = A^{-1}A^3 - 6A^{-1}A^2 + 9A^{-1}A$

$\Rightarrow \quad 4A^{-1} = A^2 - 6A + 9I$

$$\Rightarrow \quad A^{-1} = \frac{1}{4}A^2 - \frac{6}{4}A + \frac{9}{4}I$$

$$= \frac{1}{4}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \frac{6}{4}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \frac{9}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

17. (B) : For any $n \times n$ matrix A,

det (adj A) = $|A|^{n-1}$

($\because$ It holds for singular and non-singular matrices.)

Since, A is a 3×3 non-singular matrix.

So, $|\text{adj } A| = |A|^{3-1} = |A|^2$

18. (B) : Given, A is an invertible matrix of order 2.

So, $AA^{-1} = I_2 = A^{-1}A$, where I_2 is identity matrix of order 2.

$\Rightarrow \quad \det(AA^{-1}) = \det I \Rightarrow \det A \cdot \det(A^{-1}) = 1$

$$\Rightarrow \quad \det(A^{-1}) = \frac{1}{\det A}, \det A \neq 0$$

EXERCISE - 4.6

1. The given system of equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Now, $|A| = 3 - 4 = -1 \neq 0$

Hence, the system of equations is consistent.

2. The given system of equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2 + 1 = 3 \neq 0$$

Hence, the system of equations is consistent.

3. The given system of equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$$

Hence, A is singular matrix as $|A| = 0$. So, we will find (adj A)B.

$$\text{adj } A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\text{Now (adj } A)B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30-24 \\ -10+8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, the system of equations are inconsistent with no solution.

4. The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$|A| = 1(6a - 2a) - 1(4a - 2a) + 1(2a - 3a) = 4a - 2a - a = a$

Two conditions arise:

I : If $a \neq 0$, then $|A| \neq 0$, hence the system of equations is consistent and has a unique solution.

II : If $a = 0$, then $|A| = 0$. So, we need to calculate adj A.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ a & 2a \end{vmatrix} = 4a, \; C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ a & 2a \end{vmatrix} = -2a,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ a & a \end{vmatrix} = -a, \; C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ a & 2a \end{vmatrix} = -a,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ a & 2a \end{vmatrix} = a, \; C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ a & a \end{vmatrix} = 0,$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1, \; C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0,$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

$$\text{adj } A = \begin{bmatrix} 4a & -a & -1 \\ -2a & a & 0 \\ -a & 0 & 1 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A)B = \begin{bmatrix} 4a & -a & -1 \\ -2a & a & 0 \\ -a & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2a-4 \\ 0 \\ -a+4 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix} \neq O \qquad [\because a = 0]$$

Hence, the system of equations are inconsistent with no solution because if $a = 0$, then the third system of equations is not possible.

$\therefore$ $a \neq 0$ and system of equations is consistent.

5. The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$|A| = 3(0 - 5) + 1(0 + 3) - 2(0 - 6) = -15 + 3 + 12 = 0$

$\therefore$ $|A| = 0$

Hence, A is a singular matrix so, we will calculate (adj A)B.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} = -5,\ C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -3,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix} = -6,$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} = -(-10) = 10,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} = 6,$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} = -(-15+3) = 12,$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} = 1+4 = 5,\ C_{32} = (-1)^{3+2}\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} = 3,$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6$$

$$\text{adj } A = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A) \cdot B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -10 - 10 + 15 \\ -6 - 6 + 9 \\ -12 - 12 + 18 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq O$$

Hence, the system of equations is inconsistent with no solution.

6. The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$$

$= 5(18 + 10) + 1(12 - 25) + 4(-4 - 15)$

$= 140 - 13 - 76 = 51 \neq 0$

Hence, the system of equations are consistent and have a unique solution.

7. The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} = 15 - 14 = 1 \neq 0$$

$\Rightarrow$ A is non-singular matrix. So, its inverse exists.

Now, $A_{11} = (-1)^{1+1}(3) = 3$, $A_{12} = (-1)^{1+2}(7) = -7$,

$A_{21} = (-1)^{2+1}(2) = -2$, $A_{22} = (-1)^{2+2}(5) = 5$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}' = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$\text{and } A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

Solution of given system is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 12-10 \\ -28+25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Hence, $x = 2$, $y = -3$

8. The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 8 + 3 = 11 \neq 0$$

Hence, A is non-singular. So, its inverse exists.

Here, $A_{11} = (-1)^{1+1}(4) = 4$, $A_{12} = (-1)^{1+2}(3) = -3$,

$A_{21} = (-1)^{2+1}(-1) = 1, A_{22} = (-1)^{2+2}(2) = 2$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}' = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

and $A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$

Solution of given system is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} \frac{-5}{11} \\ \frac{12}{11} \end{bmatrix}$$

$$\therefore \quad x = -\frac{5}{11}, y = \frac{12}{11}$$

9. The given system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 4 & -3 \\ 3 & -5 \end{vmatrix} = -20 + 9 = -11 \neq 0$

$\Rightarrow$ A^{-1} exists and hence the given equation has a unique solution.

Here, $A_{11} = (-1)^{1+1}(-5) = -5, A_{12} = (-1)^{1+2}(3) = -3,$

$A_{21} = (-1)^{2+1}(-3) = 3, A_{22} = (-1)^{2+2}(4) = 4$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}' = \begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}$$

and $A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-11}\begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}$

Solution of given system is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$= -\frac{1}{11}\begin{bmatrix} -15+21 \\ -9+28 \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} 6 \\ 19 \end{bmatrix} = \begin{bmatrix} \frac{-6}{11} \\ \frac{-19}{11} \end{bmatrix}$$

Hence, $x = \frac{-6}{11},\ y = \frac{-19}{11}$

10. The given system of equations can be written as $AX = B$,

where $A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix} = 10 - 6 = 4 \neq 0$

$\Rightarrow$ A^{-1} exists and hence the given equation has a unique solution.

Here, $A_{11} = (-1)^{1+1}(2) = 2, A_{12} = (-1)^{1+2}(3) = -3,$

$A_{21} = (-1)^{2+1}(2) = -2, A_{22} = (-1)^{2+2}(5) = 5$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix}' = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

And $A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$

Solution of given system is given by $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$\therefore \quad x = -1, y = 4.$

11. The given system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{vmatrix} = 2\begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}$$

$= 2(10 + 3) - 1(-5) + 1(3) = 26 + 5 + 3 = 34 \neq 0.$

$\Rightarrow$ A^{-1} exists and hence the given equations have a unique solution.

Here, $A_{11} = (-1)^{1+1}\begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix} = 10 + 3 = 13,$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} = -(-5-0) = 5,$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} = 3 - 0 = 3,$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} = -(-5-3) = 8,$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 0 & -5 \end{vmatrix} = -10 - 0 = -10,$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = -(6-0) = -6,$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = -1 + 2 = 1,$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -(-2-1) = 3,$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4 - 1 = -5$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}' = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

And $A^{-1}=\frac{1}{|A|}\text{adj } A=\frac{1}{34}\begin{bmatrix}13 & 8 & 1\\ 5 & -10 & 3\\ 3 & -6 & -5\end{bmatrix}$

Solution of given system is given by $X = A^{-1}B$.

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{34}\begin{bmatrix}13 & 8 & 1\\ 5 & -10 & 3\\ 3 & -6 & -5\end{bmatrix}\begin{bmatrix}1\\ 3/2\\ 9\end{bmatrix}$$

$$=\frac{1}{34}\begin{bmatrix}13+12+9\\ 5-15+27\\ 3-9-45\end{bmatrix}=\frac{1}{34}\begin{bmatrix}34\\ 17\\ -51\end{bmatrix}=\begin{bmatrix}1\\ 1/2\\ \frac{-3}{2}\end{bmatrix}$$

Hence, $x=1, y=\frac{1}{2}$ and $z=\frac{-3}{2}$.

12. The given system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}$$

Now, $|A|=\begin{vmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{vmatrix}=1\begin{vmatrix}1 & -3\\ 1 & 1\end{vmatrix}+1\begin{vmatrix}2 & -3\\ 1 & 1\end{vmatrix}+1\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix}$

$= 1(1 + 3) + 1(2 + 3) + 1(2 - 1) = 4 + 5 + 1 = 10 \neq 0$

Hence A is non-singular matrix and so its inverse exists.

Now, $A_{11}=(-1)^{1+1}\begin{vmatrix}1 & -3\\ 1 & 1\end{vmatrix}=1+3=4,$

$A_{12}=(-1)^{1+2}\begin{vmatrix}2 & -3\\ 1 & 1\end{vmatrix}=-(2+3)=-5,$

$A_{13}=(-1)^{1+3}\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix}=2-1=1,$

$A_{21}=(-1)^{2+1}\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix}=-(-1-1)=2,$

$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}=1-1=0,$

$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}=-(1+1)=-2,$

$A_{31}=(-1)^{3+1}\begin{vmatrix}-1 & 1\\ 1 & -3\end{vmatrix}=3-1=2,$

$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 1\\ 2 & -3\end{vmatrix}=-(-3-2)=5,$

$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix}=1+2=3$

$$\text{adj } A=\begin{bmatrix}4 & -5 & 1\\ 2 & 0 & -2\\ 2 & 5 & 3\end{bmatrix}'=\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}$$

and $A^{-1}=\frac{1}{|A|}(\text{adj } A)=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}$

Solution of the system of equations is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}=\frac{1}{10}\begin{bmatrix}16+0+4\\ -20+0+10\\ 4-0+6\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}20\\ -10\\ 10\end{bmatrix}=\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}\Rightarrow x=2, y=-1 \text{ and } z=1.$$

13. The given system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}2 & 3 & 3\\ 1 & -2 & 1\\ 3 & -1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}5\\ -4\\ 3\end{bmatrix}$$

Now, $|A|=\begin{vmatrix}2 & 3 & 3\\ 1 & -2 & 1\\ 3 & -1 & -2\end{vmatrix}$

$= 2(4 + 1) - 3(-2 - 3) + 3(-1 + 6) = 10 + 15 + 15 = 40 \neq 0$

$\Rightarrow$ A^{-1} exists and hence the given equations have a unique solution.

Now, $A_{11}=(-1)^{1+1}\begin{vmatrix}-2 & 1\\ -1 & -2\end{vmatrix}=4+1=5,$

$A_{12}=(-1)^{1+2}\begin{vmatrix}1 & 1\\ 3 & -2\end{vmatrix}=-(-2-3)=5,$

$A_{13}=(-1)^{1+3}\begin{vmatrix}1 & -2\\ 3 & -1\end{vmatrix}=-1+6=5,$

$A_{21}=(-1)^{2+1}\begin{vmatrix}3 & 3\\ -1 & -2\end{vmatrix}=-(-6+3)=3,$

$A_{22}=(-1)^{2+2}\begin{vmatrix}2 & 3\\ 3 & -2\end{vmatrix}=-4-9=-13,$

$A_{23}=(-1)^{2+3}\begin{vmatrix}2 & 3\\ 3 & -1\end{vmatrix}=-(-2-9)=11,$

$A_{31}=(-1)^{3+1}\begin{vmatrix}3 & 3\\ -2 & 1\end{vmatrix}=3+6=9,$

$A_{32}=(-1)^{3+2}\begin{vmatrix}2 & 3\\ 1 & 1\end{vmatrix}=-(2-3)=1,$

$A_{33}=(-1)^{3+3}\begin{vmatrix}2 & 3\\ 1 & -2\end{vmatrix}=-4-3=-7$

$$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Solution of given system is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\Rightarrow \quad x = 1, y = 2$ and $z = -1$.

14. The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 1(12 - 5) + 1(9 + 10) + 2(-3 - 8) = 7 + 19 - 22 = 4 \neq 0$

$\Rightarrow \quad A^{-1}$ exists and hence the given equations have a unique solution.

$$\text{Now, } A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & -5 \\ -1 & 3 \end{vmatrix} = 12 - 5 = 7,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix} = -(9+10) = -19,$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -3 - 8 = -11,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} = -(-3+2) = 1,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -(-1+2) = -1,$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} = 5 - 8 = -3,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -(-5-6) = 11,$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = 4 + 3 = 7$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

Solution of given system is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\Rightarrow \quad x = 2, y = 1$ and $z = 3$.

15. $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$\Rightarrow \quad |A| = 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2)$
$= 0 - 6 + 5 = -1 \neq 0$

$\therefore \quad A^{-1}$ exists.

$$\text{Here, } A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = -4 + 4 = 0,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = -(-6+4) = 2,$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -(6-5) = -1,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -4 - 5 = -9,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -(2+3) = -5,$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 12 - 10 = 2,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = -(-8-15) = 23,$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 4 + 9 = 13$$

$$\therefore \; A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

Solution of the system of equations is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1, y = 2$ and $z = 3$.

16. Let cost of 1 kg onion = ₹ x

Cost of 1 kg wheat = ₹ y and cost of 1 kg rice = ₹ z

$\therefore$ $4x + 3y + 2z = 60,\ 2x + 4y + 6z = 90$ and $6x + 2y + 3z = 70$

This system can be written as $AX = B$, where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix}$

$= 4(12 - 12) - 3(6 - 36) + 2(4 - 24) = 90 - 40 = 50 \neq 0$

$\therefore$ A^{-1} exists.

Now, we will find adj A

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} = 0,\ A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 6 \\ 6 & 3 \end{vmatrix} = 30,$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 4 \\ 6 & 2 \end{vmatrix} = -20,\ A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = -5,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0,\ A_{23} = (-1)^{2+3}\begin{vmatrix} 4 & 3 \\ 6 & 2 \end{vmatrix} = 10,$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix} = 10,\ A_{32} = (-1)^{3+2}\begin{vmatrix} 4 & 2 \\ 2 & 6 \end{vmatrix} = -20,$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 4 & 3 \\ 2 & 4 \end{vmatrix} = 10$$

$$\text{adj } A = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

As, $AX = B \Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 6 \\ 9 \\ 7 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 0 - 45 + 70 \\ 180 + 0 - 140 \\ -120 + 90 + 70 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 25 \\ 40 \\ 40 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\therefore$ $x = 5, y = 8, z = 8$

Hence, cost of 1 kg of onion = ₹ 5, cost of 1 kg of wheat = ₹ 8 and cost of 1 kg of rice = ₹ 8

NCERT MISCELLANEOUS EXERCISE

1. Let $\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

Expanding along R_1, we get

$$= x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$$

$= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$

$= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$

$= -x^3 - x + x(\sin^2\theta + \cos^2\theta) = -x^3 - x + x(1) = -x^3$

which is independent of θ.

2. $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

$$= \cos\alpha\cos\beta\begin{vmatrix} \cos\beta & 0 \\ \sin\alpha\sin\beta & \cos\alpha \end{vmatrix} - \cos\alpha\sin\beta\begin{vmatrix} -\sin\beta & 0 \\ \sin\alpha\cos\beta & \cos\alpha \end{vmatrix} - \sin\alpha\begin{vmatrix} -\sin\beta & \cos\beta \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta \end{vmatrix}$$

$= \cos\alpha\cos\beta(\cos\beta\cos\alpha) - \cos\alpha\sin\beta(-\sin\beta\cos\alpha) - \sin\alpha(-\sin\alpha\sin^2\beta - \sin\alpha\cos^2\beta)$

$= \cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta$

$= \cos^2\alpha(\cos^2\beta + \sin^2\beta) + \sin^2\alpha(\sin^2\beta + \cos^2\beta)$

$= \cos^2\alpha + \sin^2\alpha = 1.$

3. Given $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$,

$$B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$|B| = 1(3) - 2(-1) - 2(2) = 3 + 2 - 4 = 1 \neq 0$

$\therefore$ B^{-1} exists.

Since $(AB)^{-1} = B^{-1}A^{-1}$

So, we need to calculate B^{-1}

$$B_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = 3,$$

$$B_{12} = (-1)^{1+2}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

$$B_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2,$$

$$B_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = 2,$$

$$B_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1,$$

$$B_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = 2,$$

$B_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = 6,$

$B_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = 2,$

$B_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 5$

$$\text{adj } B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\therefore \quad B^{-1} = \frac{1}{|B|}\text{adj } B = \frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Now, $(AB)^{-1} = B^{-1}A^{-1}$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

4. Now, $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$|A| = 1(15 - 1) + 2(-10 - 1) + 1(-2 - 3)$

$= 14 - 22 - 5 = -13 \neq 0$

$\therefore \quad A^{-1}$ exists.

$$\text{adj} A = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}^{\prime} = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj} A = \frac{1}{-13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$$

(i) Now, $|\text{adj} A| = \begin{vmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{vmatrix}$

$= 14(-4 - 9) - 11(-11 - 15) - 5(-33 + 20)$

$= -182 + 286 + 65 = 169 \neq 0$

$\therefore \quad \text{adj } A$ is invertible

and $(\text{adj} A)^{-1} = \dfrac{1}{|\text{adj} A|}\{\text{adj}(\text{adj} A)\}$

$$= \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} \quad \text{...(1)}$$

Also, $\text{adj}(A^{-1}) = \text{adj}\left(\dfrac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}\right)$

$$= \text{adj}\begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix} = \begin{bmatrix} -\frac{13}{169} & \frac{26}{169} & -\frac{13}{169} \\ \frac{26}{169} & -\frac{39}{169} & -\frac{13}{169} \\ -\frac{13}{169} & -\frac{13}{169} & -\frac{65}{169} \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} \quad \text{...(2)}$$

From (1) and (2), we find that $\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$

(ii) $|A^{-1}| = \left|\dfrac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}\right|$

$$= \frac{1}{(13)^3}\{-14(-4-9) + 11(-11-15) + 5(-33+20)\}$$

$$= \frac{1}{(13)^3} \times (-169) = -\frac{1}{13} \neq 0$$

$\therefore \quad (A^{-1})^{-1}$ exists and $(A^{-1})^{-1} = \dfrac{1}{|A^{-1}|}(\text{adj} A^{-1})$

$$= \frac{1}{-\frac{1}{13}} \cdot \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A. \quad \text{(Using (2))}$$

Hence, $(A^{-1})^{-1} = A$

5. The equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$

$= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$

$= 150 + 330 + 720 = 1200 \neq 0$

$\therefore \quad A^{-1}$ exists.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} = 120 - 45 = 75,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} = -(-80 - 30) = 110,$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix} = 72,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 10 \\ 9 & -20 \end{vmatrix} = 150,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 10 \\ 6 & -20 \end{vmatrix} = -100,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 0,$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 10 \\ -6 & 5 \end{vmatrix} = 15 + 60 = 75,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 10 \\ 4 & 5 \end{vmatrix} = -(10 - 40) = 30,$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} = -12 - 12 = -24$$

$$\text{adj } A = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\text{adj} A = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

As $AX = B \;\Rightarrow\; X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

Thus, $\frac{1}{x} = \frac{1}{2}, \frac{1}{y} = \frac{1}{3}, \frac{1}{z} = \frac{1}{5}$

Hence, $x = 2, y = 3, z = 5$.

6. (A) : $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

$|A| = xyz \neq 0$, A^{-1} exists.

Now, $A_{11} = (-1)^{1+1}\begin{vmatrix} y & 0 \\ 0 & z \end{vmatrix} = yz,$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = 0,$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & y \\ 0 & 0 \end{vmatrix} = 0,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = 0,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} x & 0 \\ 0 & z \end{vmatrix} = xz,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = 0,$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 0 \\ y & 0 \end{vmatrix} = 0,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = 0,$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix} = xy$$

$$\therefore \quad A^{-1} = \frac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

7. (D) : $|A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

Expanding along R_1, we get

$= 1(1 + \sin^2\theta) - \sin\theta\,(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$

$= 1 + \sin^2\theta + 1 + \sin^2\theta = 2(1 + \sin^2\theta)$

As, $\sin^2\theta \in [0, 1]$

$\Rightarrow \quad 1 + \sin^2\theta \in [1, 2]$

$\Rightarrow \; 2(1 + \sin^2\theta) \in [2, 4]$