Determinants

CHAPTER 4

TRY YOURSELF — SOLUTIONS

1. (i) We have, $\begin{vmatrix} 1/2 & 8 \\ 4 & 2 \end{vmatrix} = \frac{1}{2}(2) - 8(4) = 1 - 32 = -31$

(ii) We have, $\begin{vmatrix} \cos 90^\circ & -\cos 45^\circ \\ \sin 90^\circ & \sin 45^\circ \end{vmatrix}$

$= \cos 90^\circ \sin 45^\circ + \cos 45^\circ \sin 90^\circ$

$= \sin(90^\circ + 45^\circ) = \sin 135^\circ = \sin(180^\circ - 45^\circ)$

$= \sin 45^\circ = \frac{1}{\sqrt{2}}$

2. We have, $\begin{vmatrix} 3x & 7 \\ 2 & 4 \end{vmatrix} = 10$

$\Rightarrow$ $(3x)(4) - 7(2) = 10 \Rightarrow 12x = 24 \Rightarrow x = 2$

3. We have, $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$

$= 1(1 - 0) = 1$

4. We have, $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix} = 8$

$\Rightarrow$ $x(-x^2 - 1) - \sin\theta(-\sin\theta \cdot x - \cos\theta)$
$+ \cos\theta(-\sin\theta + x\cos\theta) = 8$

$\Rightarrow -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta = 8$

$\Rightarrow$ $-x^3 - x + x(\sin^2\theta + \cos^2\theta) = 8$

$\Rightarrow$ $-x^3 - x + x = 8 \Rightarrow x^3 + 8 = 0$

$\Rightarrow$ $(x + 2)(x^2 - 2x + 4) = 0$

$\Rightarrow$ $x + 2 = 0$ $\quad [\because x^2 - 2x + 4 > 0\ \forall\ x]$

$\Rightarrow$ $x = -2$

5. Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$

$= \frac{1}{2}\left[at_1^2(2at_2 - 2at_3) - at_2^2(2at_1 - 2at_3) + at_3^2(2at_1 - 2at_2)\right]$

$= \frac{1}{2}\left[2a^2t_1^2t_2 - 2a^2t_1^2t_3 - 2a^2t_1t_2^2 + 2a^2t_2^2t_3 + 2a^2t_1t_3^2 - 2a^2t_2t_3^2\right]$

$= a^2\left[t_1^2(t_2 - t_3) - t_1(t_2^2 - t_3^2) + t_2t_3(t_2 - t_3)\right]$

$= a^2(t_2 - t_3)\left[t_1^2 - t_1t_2 - t_1t_3 + t_2t_3\right]$

$= a^2(t_2 - t_3)(t_1(t_1 - t_2) - t_3(t_1 - t_2))$

$= a^2(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)$

6. The given points are collinear, iff

$\begin{vmatrix} 2 & -5 & 1 \\ -4 & 5 & 1 \\ x & 15 & 1 \end{vmatrix} = 0$

$\Rightarrow$ $2(5 - 15) + 5(-4 - x) + 1(-60 - 5x) = 0$

$\Rightarrow$ $-20 - 20 - 5x - 60 - 5x = 0$

$\Rightarrow$ $-10x = 100 \Rightarrow x = -10$

7. Given points are $A(1, 2)$, $B(3, 8)$. Let $P(x, y)$ be any point on the line containing $A(1, 2)$ and $B(3, 8)$.

$\Rightarrow$ $P(x, y)$, $A(1, 2)$, $B(3, 8)$ are collinear.

$\Rightarrow$ $\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 8 & 1 \end{vmatrix} = 0 \Rightarrow x(2 - 8) - y(1 - 3) + 1(8 - 6) = 0$

$\Rightarrow$ $-6x + 2y + 2 = 0 \Rightarrow 3x - y - 1 = 0$, which is the required equation of line.

8. (i) We have, $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$

Since, minor of a_{ij} is M_{ij}.

$\therefore$ M_{11} = Minor of $a_{11} = -1$

M_{12} = Minor of $a_{12} = 0$

M_{21} = Minor of $a_{21} = 20$

M_{22} = Minor of $a_{22} = 5$

Also, let co-factor of a_{ij} is A_{ij}

A_{11} = Co-factor of $a_{11} = (-1)^{1+1}M_{11} = (-1)^2(-1) = -1$

A_{12} = Co-factor of $a_{12} = (-1)^{1+2}M_{12} = (-1)^3(0) = 0$

A_{21} = Co-factor of $a_{21} = (-1)^{2+1}M_{21} = (-1)^3(20) = -20$

A_{22} = Co-factor of $a_{22} = (-1)^{2+2}M_{22} = (-1)^4(5) = 5$

(ii) We have, $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

Since, minor of a_{ij} is M_{ij}.

$\therefore$ $M_{11} = \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = -2 - 10 = -12$

$M_{12} = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = 8 - 6 = 2$

$M_{13} = \begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix} = 20 + 3 = 23$

$M_{21} = \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = -6 - 10 = -16$

$M_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 2 - 6 = -4$

$M_{23} = \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} = 5 + 9 = 14$

$M_{31} = \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} = -6 + 2 = -4$

$M_{32} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 2 - 8 = -6$

$M_{33} = \begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} = -1 + 12 = 11$

Also, let the co-factor of a_{ij} is A_{ij}

$\therefore \quad A_{11} = (-1)^{1+1}M_{11} = (-1)^2(-12) = -12$

$A_{12} = (-1)^{1+2}M_{12} = (-1)^3\, 2 = -2$

$A_{13} = (-1)^{1+3}M_{13} = (-1)^4 23 = 23$

$A_{21} = (-1)^{2+1}M_{21} = (-1)^3(-16) = 16$

$A_{22} = (-1)^{2+2}M_{22} = (-1)^4(-4) = -4$

$A_{23} = (-1)^{2+3}M_{23} = (-1)^5(14) = -14$

$A_{31} = (-1)^{3+1}M_{31} = (-1)^4(-4) = -4$

$A_{32} = (-1)^{3+2}M_{32} = (-1)^5(-6) = 6$

$A_{33} = (-1)^{3+3}M_{33} = (-1)^6 11 = 11$

9. Let $\Delta = \begin{vmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{vmatrix}$

Let M_{ij} and A_{ij} respectively be the minor and cofactor of a_{ij} in Δ.

$M_{11} = \begin{vmatrix} 5 & 0 \\ 7 & 1 \end{vmatrix} = 5 - 0 = 5; A_{11} = (-1)^{1+1}M_{11} = (-1)^2(5) = 5$

$M_{21} = \begin{vmatrix} 2 & 6 \\ 7 & 1 \end{vmatrix} = 2 - 42 = -40; A_{21} = (-1)^{2+1}M_{21} = (-1)^3(-40) = 40$

$M_{31} = \begin{vmatrix} 2 & 6 \\ 5 & 0 \end{vmatrix} = 0 - 30 = -30; A_{31} = (-1)^{3+1}M_{31} = (-1)^4(-30) = -30$

Now, det. $\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

$= 0(5) + 1(40) + 3(-30)$

$= 40 - 90 = -50$

10. We have, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$ (Given)

$\Rightarrow$ A is invertible

Now, $A_{11} = d, A_{12} = -c, A_{21} = -b, A_{22} = a$

$\therefore \quad \text{adj } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}' = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Hence, $A^{-1} = \frac{1}{|A|} \text{adj } A$

$= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

11. By reversal law, we have

$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} A^{-1}$ ...(i)

To find A^{-1} : we have, $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

$\Rightarrow \quad |A| = 6 - 5 = 1 \neq 0$

$\therefore \quad A^{-1}$ exists.

Now, $A_{11} = 3, A_{12} = -5, A_{21} = -1, A_{22} = 2$

$\therefore \quad A^{-1} = \frac{1}{|A|} \text{adj } A = 1\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

From (i), $(AB)^{-1} = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 12-25 & -4+10 \\ 9-20 & -3+8 \end{bmatrix}$

$= \begin{bmatrix} -13 & 6 \\ -11 & 5 \end{bmatrix}$

12. Given, $A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

i.e., $AB = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, where $B = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ ...(i)

Now, $|B| = \begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} = 4 + 2 = 6 \neq 0$

$\therefore \quad B^{-1}$ exists.

$\therefore \quad B^{-1} = \frac{1}{|B|} \text{adj } B = \frac{1}{6}\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}' = \frac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

Post multiplying both sides of (i) by B^{-1}, we get

$A(B \cdot B^{-1}) = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} B^{-1}$

$\Rightarrow \quad AI = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \quad [\because BB^{-1} = I]$

$\Rightarrow \quad A = \frac{1}{6}\begin{bmatrix} 24 & 12 \\ -6 & 6 \end{bmatrix} \quad \therefore \quad A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

13. We have, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

Now, $|A| = 1(0-2) - 1(1-6) + 1(1)$

$= -2 + 5 + 1 = 4 \neq 0 \quad \therefore A^{-1}$ exists

Now, $A_{11} = -2, A_{12} = 5, A_{13} = 1, A_{21} = 0, A_{22} = -2, A_{23} = 2,$
$A_{31} = 2, A_{32} = -1, A_{33} = -1$

$$\text{So, adj } A = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

The given system of equations can be written as $AX = B$ *i.e.*,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$\because \quad A^{-1}$ exists.

So, the system has a unique solution given by $X = A^{-1}B$.

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -12+0+24 \\ 30-14-12 \\ 6+14-12 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore \quad x = 3, y = 1, z = 2.$

14. The given system of equations can be written as

$$\begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix} \text{ i.e., } AX = B,$$

$$\text{where } A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{vmatrix} = 4(6-6) - 3(3-18) + 2(2-12)$$

$= 0 + 45 - 20 = 25 \neq 0$

$\therefore \quad A^{-1}$ exists.

So, the system of equations has a unique solution and its solution is given by $X = A^{-1}B$.

Now, $A_{11} = 0, A_{12} = 15, A_{13} = -10, A_{21} = -5, A_{22} = 0,$
$A_{23} = 10, A_{31} = 5, A_{32} = -10, A_{33} = 5.$

$$\text{So, adj } A = \begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}\begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 0-225+350 \\ 900+0-700 \\ -600+450+350 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 125 \\ 200 \\ 200 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\therefore \quad x = 5, y = 8, z = 8.$