Continuity and Differentiability

CHAPTER 5

EXAM DRILL

SOLUTIONS

1. (d) : Given $f(x)$ is continuous at every point of its domain.

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

$\Rightarrow 5 \times 1 - 4 = 4 \times 1 + 3a$

$\Rightarrow 4 + 3a = 1 \Rightarrow 3a = -3 \Rightarrow a = -1$

2. (a) : Here $y = a \sin mx + b\cos mx$ ...(i)

$\Rightarrow \frac{dy}{dx} = ma\cos mx - mb\sin mx$

$\Rightarrow \frac{d^2y}{dx^2} = -m^2[a\sin mx + b\cos mx] = -m^2y$ [From (i)]

3. (c) : We have given,

$$f(x) = \frac{4-x^2}{4x-x^3} = \frac{(4-x^2)}{x(4-x^2)}$$

$$= \frac{(4-x^2)}{x(2^2-x^2)} = \frac{4-x^2}{x(2+x)(2-x)}$$

Since, $f(x)$ is not defined at $x = 0, -2$ and 2.

$\therefore f(x)$ is discontinuous at exactly three points $x = 0, x = -2$ and $x = 2$.

4. (b) : We have given, $y = \log\left(\frac{1-x^2}{1+x^2}\right)$

Taking derivative w.r.t. 'x' on both sides, we get

$$\frac{dy}{dx} = \frac{1}{\left(\frac{1-x^2}{1+x^2}\right)} \cdot \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{(1+x^2)}{(1-x^2)} \cdot \frac{(1+x^2)\cdot(-2x)-(1-x^2)\cdot 2x}{(1+x^2)^2}$$

$$= \frac{-2x[1+x^2+1-x^2]}{(1-x^2)\cdot(1+x^2)} = \frac{-4x}{1-x^4} \quad [\because a^2-b^2 = (a+b)(a-b)]$$

5. (a) : Given, f is continuous on $[-2, 2]$.

$$\text{L.H.L.} = \lim_{x\to 0}\frac{\sqrt{1+cx}-\sqrt{1-cx}}{x} = \lim_{x\to 0}\frac{2cx}{x[\sqrt{1+cx}+\sqrt{1-cx}]} = c$$

$$\text{R.H.L.} = \lim_{x\to 0}\frac{x+3}{x+1} = 3$$

$\therefore c = 3$

6. (b) : $y = f(x^2+2) \Rightarrow \frac{dy}{dx} = f'(x^2+2)\times 2x$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\text{at } x=1} = f'(3)\times 2 = 5\times 2 = 10$$

7. (b) : $y = e^x \cdot e^{x^2} \cdot e^{x^3} \cdot e^{x^4} \ldots e^{x^n} \ldots$

$y = e^{x+x^2+x^3+x^4+\ldots+x^n+\ldots}$

$y = e^{\frac{x}{1-x}}$ $[\because 0 < x < 1]$

$$\therefore \frac{dy}{dx} = e^{\frac{x}{1-x}} \times \frac{(1-x)(1)-x(-1)}{(1-x)^2} = e^{\frac{x}{1-x}}\frac{1}{(1-x)^2}$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=\frac{1}{2}} = e^{\frac{1/2}{1-1/2}}\left(\frac{1}{\left(1-\frac{1}{2}\right)^2}\right) = 4e$$

8. (a) : Let $y = \sin(x^3)$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \cos(x^3)\cdot 3x^2 = 3x^2\cos(x^3)$ ...(i)

Let $z = \cos(x^3)$

Differentiating w.r.t. x, we get

$\frac{dz}{dx} = -\sin(x^3)\cdot 3x^2 = -3x^2\sin(x^3)$ ...(ii)

Dividing (i) by (ii), we get $\frac{dy}{dz} = \frac{3x^2\cos(x^3)}{-3x^2\sin(x^3)} = -\cot x^3$.

9. (a) : From the given graph, the function is continuous.

The given graph is that of $y = |x-1|$

Now, $f(x) = y = \begin{cases} x-1, & x \ge 1 \\ 1-x, & x < 1 \end{cases}$

$$Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{(1+h-1)-0}{h} = 1$$

$$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{(1-1+h)-0}{-h} = -1$$

Since, $Rf'(1) \ne Lf'(1)$

So, the given function is not differentiable.

10. Given, $f(x) = |\cos x|$

We have to find out $f'\left(\frac{\pi}{4}\right)$

Since, $\cos x > 0$ when $x \in \left[0, \frac{\pi}{2}\right]$

$\therefore$ $f(x) = \cos x$

$\Rightarrow$ $f'(x) = -\sin x$

$$\Rightarrow f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$$

11. The greatest integer function defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at $x = 1$.

12. $\frac{d}{dx}(a^x - x^x) = a^x \log a - x^x(1 + \log x)$

$$\left[\because \frac{d}{dx}(a^x) = a^x \log a \text{ and } \frac{d}{dx}(x^x) = x^x(1 + \log x)\right]$$

13. Here, $y = \cos x$

$\therefore$ $\frac{d^3y}{dx^3} = 1$

$$\left[\because \frac{dy}{dx} = -\sin x; \frac{d^2y}{dx^2} = -\cos x; \frac{d^3y}{dx^3} = \sin x\right]$$

14. A constant function is a continuous function.

15. $\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$

16. Given $f(x) = \frac{\sin 10x}{x}, x \neq 0$ is continuous at $x = 0$.

$$\therefore \lim_{x\to 0} f(x) = f(0) \Rightarrow \lim_{x\to 0}\frac{\sin 10x}{x} = f(0)$$

$$\Rightarrow 10 \lim_{10x\to 0}\frac{\sin 10x}{10x} = f(0) \Rightarrow f(0) = 10$$

17. Here, $y = f(\log_e x)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\{f(\log_e x)\} = f'(\log_e x) \times \frac{1}{x}$$

Thus, $\left.\frac{dy}{dx}\right|_{x=e} = \frac{f'(\log_e e)}{e} = \frac{f'(1)}{e} = \frac{2}{e}$ $[\because f'(1) = 2 \text{ (Given)}]$

18. We have, $f(x) = \begin{cases} \frac{kx}{|x|}, & x < 0 \\ 3, & x \geq 0 \end{cases}$

L.H.L. $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0}\frac{kx}{-x} = -k$

R.H.L. $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3 = 3$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ L.H.L. = R.H.L. $= f(0)$

$\Rightarrow$ $-k = 3 \Rightarrow k = -3$

19. Given, $f(x) = \sin^{-1}x + \cos^{-1}x + 2$

$$\Rightarrow f(x) = \frac{\pi}{2} + 2 \qquad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$\Rightarrow$ $f'(x) = 0 \Rightarrow f'(1) = 0$

20. **(i) (c)** **(ii) (b)** **(iii) (b)**
(iv) (b) **(v) (a)**

21. **(i)** We have $f(x) = (\cos x + i \sin x)(\cos 2x + i \sin 2x)$ $(\cos 3x + i \sin 3x) \ldots \cos(nx + i \sin x)$

$y = f(x) = e^{ix} \cdot e^{2ix} \cdot e^{3ix} \ldots.. e^{nix}$.

$$y = e^{(1+2+3+\ldots n)ix} = e^{\left\{\frac{n(n+1)}{2}\right\}ix}$$

$$\text{Now, } y' = e^{\left\{\frac{n(n+1)}{2}\right\}ix} \cdot \frac{in(n+1)}{2}$$

$$y'' = -\left\{\frac{n(n+1)}{2}\right\}^2 \cdot e^{\frac{n(n+1)}{2}ix}$$

$$y'' = -\left\{\frac{n(n+1)}{2}\right\}^2 \cdot y$$

$$y''(1) = -\left\{\frac{n(n+1)}{2}\right\}^2 \cdot y(1)$$

$$y''(1) = f''(1) = -\left\{\frac{n(n+1)}{2}\right\}^2 \qquad [\because y(1) = 1]$$

(ii) We have $y = f(x) = e^x \sin x + 2 \log \sin x$

Differentiate w.r.t. x, we get

$$y' = e^x \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(e^x) + 2\frac{d}{dx}(\log \sin x)$$

$$= e^x \cos x + e^x \sin x + 2 \times \frac{1}{\sin x} \times \cos x$$

$$= e^x \cos x + e^x \sin x + 2\cot x$$

Again, differentiate w.r.t. x, we get

$$y'' = e^x \cdot (-\sin x) + \cos x \cdot e^x + e^x \cos x + \sin x \cdot e^x + 2(-\text{cosec}^2 x)$$

$$y'' = -\sin x \cdot e^x + \cos x \cdot e^x + \cos x \cdot e^x + \sin x \cdot e^x - 2\text{cosec}^2 x$$

$$= 2\cos x \cdot e^x - 2\,\text{cosec}^2 x$$

22. Here, $x = f(t)$, $y = g(t)$

$$\Rightarrow \frac{dx}{dt} = f'(t), \ \frac{dy}{dt} = g'(t)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{f'(t)\,g''(t) - g'(t)\,f''(t)}{[f'(t)]^2} \times \frac{dt}{dx}$$

$$= \frac{f'(t)\,g''(t) - g'(t)\,f''(t)}{[f'(t)]^3}$$

23. Let $y = (\sin x^2)^{\log x^2}$. Then, $y = e^{\log x^2 \cdot \log \sin x^2}$ $[\because a^b = e^{b \log a}]$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{\log x^2 . \log \sin x} \frac{d}{dx}\{\log x^2 . \log \sin x^2\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x^2)^{\log x^2}\left\{\log \sin x^2 \frac{d}{dx}(\log x^2) + \log x^2 \frac{d}{dx}(\log \sin x^2)\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x^2)^{\log x^2}\left\{\frac{2x \log \sin x^2}{x^2} + \log x^2 \times \frac{2x}{\sin x^2} \times \cos x^2\right\}$$

$$\Rightarrow \frac{dy}{dx} = 2x(\sin x^2)^{\log x^2}\left\{\frac{\log \sin x^2}{x^2} + \cot x^2 \cdot \log x^2\right\}$$

24. Given, $y = \cos^{-1}(2x) + 2\cos^{-1}\sqrt{1-4x^2}$

Put $2x = \sin\theta$

Thus, $y = \cos^{-1}(\sin\theta) + 2\cos^{-1}\sqrt{1-\sin^2\theta}$

$$= \cos^{-1}\left(\cos\left(\frac{\pi}{2}-\theta\right)\right) + 2\cos^{-1}(\cos\theta)$$

$$= \frac{\pi}{2} - \theta + 2\theta = \frac{\pi}{2} + \theta = \frac{\pi}{2} + \sin^{-1}(2x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-4x^2}}\frac{d}{dx}(2x) = \frac{2}{\sqrt{1-4x^2}}$$

OR

Here, $y = f(e^x)\, e^{f(x)}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\{f(e^x)e^{f(x)}\} = f'(e^x)\frac{d}{dx}(e^x)\, e^{f(x)} + f(e^x)\, e^{f(x)}\frac{d}{dx}(f(x))$$

$$\Rightarrow \frac{dy}{dx} = f'(e^x)\, e^x \cdot e^{f(x)} + f(e^x)e^{f(x)} f'(x)$$

Now $\left.\frac{dy}{dx}\right|_{x=0} = f'(1)\cdot 1\cdot e^{f(0)} + f(1)\, e^{f(0)} f'(0)$

$= 2\cdot 1 + 0 = 2$

25. Let $y = \sin(e^{x^2})$. Putting $x^2 = v$ and $u = e^{x^2} = e^v$, we get

$y = \sin u$, where $u = e^v$ and $\frac{dv}{dx} = 2x$.

$$\therefore \quad \frac{dy}{du} = \cos u, \frac{du}{dv} = e^v \text{ and } \frac{dv}{dx} = 2x.$$

Now, $\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx} = \cos u \times e^v \times 2x = \cos(e^v)\times e^v \times 2x \qquad [\because u = e^v]$$

$$\Rightarrow \frac{dy}{dx} = \cos(e^{x^2})\times e^{x^2}\times 2x \qquad [\because v = x^2]$$

26. We know that a polynomial function is differentiable everywhere. Therefore, $f(x)$ is differentiable at $x = 3$.

$$\therefore \quad f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$

$$\Rightarrow \quad f'(3) = \lim_{h\to 0}\frac{\{(3+h)^2+2(3+h)+7\}-\{9+6+7\}}{h}$$

$$\Rightarrow \quad f'(3) = \lim_{h\to 0}\frac{8h+h^2}{h} = \lim_{h\to 0}(8+h) = 8.$$

27. We have given,

$$f(x) = \begin{cases} 3x+5, & \text{if } x \ge 2 \\ x^2, & \text{if } x < 2\end{cases}$$

At $x = 2$,

L.H.L. $= \lim_{x\to 2^-}(x)^2$

$= \lim_{h\to 0}(2-h)^2 = \lim_{h\to 0}(4+h^2-4h) = 4$

and R.H.L. $= \lim_{x\to 2^+}(3x+5)$

$= \lim_{h\to 0}[3(2+h)+5] = 6+5 = 11$

Since, L.H.L. ≠ R.H.L. at $x = 2$

Therefore, $f(x)$ is discontinuous at $x = 2$.

28. Let $y = 2^{\cos^2 x}$

Taking log on both sides, we get

$\log y = \log 2^{\cos^2 x} = \cos^2 x \cdot \log 2$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \log 2\frac{d}{dx}(\cos x)^2$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \log 2\cdot[2\cos x]\cdot\frac{d}{dx}(\cos x)$$

$= \log 2 \cdot 2\cos x \cdot (-\sin x) = \log 2 \cdot [-(\sin 2x)]$

$$\therefore \frac{dy}{dx} = -y\cdot(\sin 2x)\log 2 = -2^{\cos^2 x}\cdot(\sin 2x)\log 2$$

29. Here, $x = \sin\left(\frac{1}{a}\log y\right)$

$$\Rightarrow \quad \sin^{-1}x = \frac{1}{a}\log y \Rightarrow \log y = a\sin^{-1}x$$

$$\Rightarrow \quad y = e^{a\sin^{-1}x}$$

Differentiating both sides w.r.t. x, we get

$$y_1 = e^{a\sin^{-1}x}\times\frac{a}{\sqrt{1-x^2}} \Rightarrow y_1 = \frac{ay}{\sqrt{1-x^2}}$$

Differentiating again w.r.t. x, we get

$$y_2 = \frac{\sqrt{1-x^2}\times ay_1 - ay\times\frac{1}{2}\times\frac{1}{\sqrt{1-x^2}}\times(-2x)}{(1-x^2)}$$

$$\Rightarrow \quad y_2 = \frac{(1-x^2)ay_1 + axy}{(1-x^2)\sqrt{1-x^2}}$$

$$\Rightarrow \quad y_2 = \frac{ay_1}{\sqrt{1-x^2}} + \frac{xay}{(1-x^2)\sqrt{1-x^2}}$$

$$\Rightarrow \quad y_2 = \frac{a^2y}{1-x^2} + \frac{xy_1}{(1-x^2)} \Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0$$

30. Let $u=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v=\tan^{-1}x$.

Putting $x = \tan\theta$, we get

$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)$

$\Rightarrow\quad u = 2\theta = 2\tan^{-1}x$

$$\left[\because\ -1<x<1\ \Rightarrow -\frac{\pi}{4}<\theta<\frac{\pi}{4}\Rightarrow -\frac{\pi}{2}<2\theta<\frac{\pi}{2}\right]$$

Thus, we have

$u = 2\tan^{-1}x$ and $v = \tan^{-1}x$

$\Rightarrow\quad \frac{du}{dx}=\frac{2}{1+x^2}$ and $\frac{dv}{dx}=\frac{1}{1+x^2}$

$\therefore\quad \frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{2/1+x^2}{1/1+x^2}=2$

31. The given series may be written as

$y=a^{(x^y)}$

$\Rightarrow\quad \log y = x^y \log a$ [Taking log on both sides]

$\Rightarrow\quad \log(\log y) = y\log x + \log(\log a)$

[Taking log on both sides]

$\Rightarrow\quad \frac{1}{\log y}\frac{d}{dx}(\log y)=\frac{dy}{dx}\log x+y\frac{d}{dx}(\log x)+0$

[Differentiating both sides w.r.t. x]

$\Rightarrow\quad \frac{1}{\log y}\frac{1}{y}\frac{dy}{dx}=\frac{dy}{dx}\log x+y\times\frac{1}{x}$

$\Rightarrow\quad \frac{dy}{dx}\left\{\frac{1}{y\log y}-\log x\right\}=\frac{y}{x}$

$\Rightarrow\quad \frac{dy}{dx}\left\{\frac{1-y\log y\log x}{y\log y}\right\}=\frac{y}{x}$

$\Rightarrow\quad \frac{dy}{dx}=\frac{y^2\log y}{x(1-y\log y\log x)}$

32. Let $z=\frac{2x-1}{x^2+1}$.

Then, $y=f(z)$

$\Rightarrow\quad \frac{dy}{dx}=\frac{d}{dx}\{f(z)\}=\frac{d}{dz}\{f(z)\}.\frac{dz}{dx}$

$\Rightarrow\quad \frac{dy}{dx}=f'(z)\frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$

$\Rightarrow\quad \frac{dy}{dx}=f'(z)\left\{\frac{2(x^2+1)-(2x-1)2x}{(x^2+1)^2}\right\}$

$\Rightarrow\quad \frac{dy}{dx}=(\sin z^2)\frac{2(x^2+1)-(4x^2-2x)}{(x^2+1)^2}$

$[\because\ f'(x)=\sin x^2\ \therefore\ f'(z)=\sin z^2]$

$\Rightarrow\quad \frac{dy}{dx}=2\sin\left(\frac{2x-1}{x^2+1}\right)^2\left\{\frac{1+x-x^2}{(x^2+1)^2}\right\}$

OR

We have, $y=\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)$

$\Rightarrow\quad y=\sin^{-1}\left(x\sqrt{1-(\sqrt{x})^2}-\sqrt{x}\sqrt{1-x^2}\right)$

$\Rightarrow\quad y=\sin^{-1}x-\sin^{-1}\sqrt{x}$

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{d}{dx}(\sqrt{x})$

$=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2}\cdot\frac{1}{\sqrt{x-x^2}}$

33. Let $y=\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}$

$\therefore\quad \frac{dy}{dx}=\frac{d}{dx}\left(\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}\right)$

$=\frac{(e^{2x}-e^{-2x})(2e^{2x}-2e^{-2x})-(e^{2x}+e^{-2x})(2e^{2x}+2e^{-2x})}{(e^{2x}-e^{-2x})^2}$

$=\frac{(2e^{4x}-2-2+2e^{-4x})-(2e^{4x}+2+2+2e^{-4x})}{(e^{2x}-e^{-2x})^2}$

$=\frac{2e^{4x}-4+2e^{-4x}-2e^{4x}-4-2e^{-4x}}{(e^{2x}-e^{-2x})^2}$

$=\frac{-8}{(e^{2x}-e^{-2x})^2}$

34. Let $t = \tan\theta$.

Then, $t>1\Rightarrow \tan\theta>1\ \Rightarrow\frac{\pi}{4}<\theta<\frac{\pi}{2}\Rightarrow\frac{\pi}{2}<2\theta<\pi$

$\therefore\quad x=\sin^{-1}\left\{\frac{2t}{1+t^2}\right\}=\sin^{-1}\left\{\frac{2\tan\theta}{1+\tan^2\theta}\right\}$

$\Rightarrow\quad x = \sin^{-1}(\sin 2\theta) = \sin^{-1}\{\sin(\pi-2\theta)\} = \pi-2\theta$

$= \pi - 2\tan^{-1}t$

$\Rightarrow\quad \frac{dx}{dt}=0-\frac{2}{1+t^2}=\frac{-2}{1+t^2}$

and, $y=\tan^{-1}\left\{\frac{2t}{1-t^2}\right\}$

$\Rightarrow\quad y=\tan^{-1}\left\{\frac{2\tan\theta}{1-\tan^2\theta}\right\}=\tan^{-1}(\tan 2\theta)$

$=\tan^{-1}\{-\tan(\pi-2\theta)\}$

$\Rightarrow\quad y = -\tan^{-1}\{\tan(\pi-2\theta)\} = -(\pi-2\theta) = -\pi+2\tan^{-1}t$

$\Rightarrow\quad \frac{dy}{dt}=0+\frac{2}{1+t^2}=\frac{2}{1+t^2}$

$\therefore\quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{2}{1+t^2}}{-\frac{2}{1+t^2}}=-1.$

35. We have given,

$$f(x)=\begin{cases}\dfrac{2x^2-3x-2}{x-2}, & \text{if } x\neq 2\\ 5, & \text{if } x=2\end{cases}$$

$$\text{At } x=2,\ \text{L.H.L.}=\lim_{x\to 2^-}\frac{2x^2-3x-2}{x-2}$$

$$=\lim_{h\to 0}\frac{2(2-h)^2-3(2-h)-2}{(2-h)-2}$$

$$=\lim_{h\to 0}\frac{8+2h^2-8h-6+3h-2}{-h}$$

$$=\lim_{h\to 0}-(2h-5)=5$$

$$\text{and R.H.L.}=\lim_{x\to 2^+}\frac{2x^2-3x-2}{x-2}$$

$$=\lim_{h\to 0}\frac{2(2+h)^2-3(2+h)-2}{(2+h)-2}$$

$$=\lim_{h\to 0}\frac{8+2h^2+8h-6-3h-2}{h}$$

$$=\lim_{h\to 0}(2h+5)=5$$

also, $f(2)=5$ (given)
Hence, L.H.L. = R.H.L. = $f(2)$
Therefore, $f(x)$ is continuous at $x=2$.

OR

We have given,

$$f(x)=\begin{cases}|x|\cos\dfrac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

$$\text{At } x=0,\ \text{L.H.L.}=\lim_{x\to 0^-}|x|\cos\frac{1}{x}$$

$$=\lim_{h\to 0}|0-h|\cos\frac{1}{0-h}$$

$$=\lim_{h\to 0}h\cos\left(\frac{-1}{h}\right)$$

$= 0\times$ [an oscillating number between –1 and 1] = 0

$$\text{and R.H.L.}=\lim_{x\to 0^+}|x|\cos\frac{1}{x}$$

$$=\lim_{h\to 0}|0+h|\cos\frac{1}{(0+h)}$$

$$=\lim_{h\to 0}h\cos\frac{1}{h}$$

$= 0\times$[an oscillating number between –1 and 1] = 0
Also, $f(0)=0$ [Given]
Hence, L.H.L. = R.H.L. = $f(0)$
Therefore, $f(x)$ is continuous at $x=0$.

36. We have given,

$$f(x)=\begin{cases}x^2\sin\dfrac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

For differentiability at $x=0$,

$$Lf'(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x^2\sin\frac{1}{x}-0}{x-0}$$

$$=\lim_{h\to 0}\frac{(0-h)^2\sin\left(\frac{1}{0-h}\right)}{0-h}=\lim_{h\to 0}\frac{h^2\sin\left(\frac{-1}{h}\right)}{-h}$$

$$=\lim_{h\to 0}h\sin\left(\frac{1}{h}\right)\qquad[\because \sin(-x)=-\sin x]$$

$= 0\times$ [an oscillating number between –1 and 1] = 0

$$Rf'(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x^2\sin\frac{1}{x}-0}{x-0}$$

$$=\lim_{h\to 0}\frac{(0+h)^2\sin\left(\frac{1}{0+h}\right)}{0+h}=\lim_{h\to 0}\frac{h^2\sin(1/h)}{h}$$

$$=\lim_{h\to 0}h\sin(1/h)$$

$= 0\times$ [an oscillating number between –1 and 1] = 0
$[\because -1\le\sin x\le 1\ \forall\ x\in R]$

$\because\quad Lf'(0)=Rf'(0)$
Therefore, $f(x)$ is differentiable at $x=0$.

37. We have given, $y^x=e^{y-x}$

$\Rightarrow\quad \log y^x=\log e^{y-x}$

$\Rightarrow\quad x\log y=(y-x)\cdot\log e=(y-x)$

$$\Rightarrow \log y=\frac{(y-x)}{x}\qquad\text{...(i)}$$

On differentiating (i) w.r.t. x, we get

$$\frac{d}{dy}(\log y)\cdot\frac{dy}{dx}=\frac{d}{dx}\frac{(y-x)}{x}$$

$$\Rightarrow\ \frac{1}{y}\cdot\frac{dy}{dx}=\frac{x\cdot\frac{d}{dx}(y-x)-(y-x)\cdot\frac{d}{dx}\cdot x}{x^2}$$

[By quotient rule]

$$\Rightarrow\ \frac{1}{y}\frac{dy}{dx}=\frac{x\left(\frac{dy}{dx}-1\right)-(y-x)}{x^2}$$

$$\Rightarrow\ \frac{x^2}{y}\cdot\frac{dy}{dx}=x\frac{dy}{dx}-x-y+x$$

$$\Rightarrow\ \frac{dy}{dx}\left(\frac{x^2}{y}-x\right)=-y$$

$$\therefore\ \frac{dy}{dx}=\frac{-y^2}{x^2-xy}=\frac{y^2}{x(y-x)}=\frac{y^2}{x^2}\cdot\frac{1}{\frac{(y-x)}{x}}$$

$$=\frac{(1+\log y)^2}{\log y}\left[\because \log y=\frac{y-x}{x},\ \text{from (i)}\Rightarrow \log y=\frac{y}{x}-1\ \Rightarrow 1+\log y=\frac{y}{x}\right]$$