Continuity and Differentiability

CHAPTER 5

NCERT FOCUS

SOLUTIONS

EXERCISE - 5.1

1. We have, $f(x) = 5x - 3$

At $x = 0$

We have $f(0) = -3$

L.H.L. = $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} 5(0-h)-3 = -3$

R.H.L. = $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} 5(0+h)-3 = -3$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\therefore$ f is continuous at $x = 0$

At $x = -3$

We have, $f(-3) = 5(-3) - 3 = -18$

L.H.L. = $\lim_{x\to -3^-} f(x) = \lim_{h\to 0} [5(-3-h)-3]$

$= \lim_{h\to 0} (-15-5h-3) = -18$

R.H.L. = $\lim_{x\to -3^+} f(x) = \lim_{h\to 0} [5(-3+h)-3]$

$= \lim_{h\to 0} (-15+5h-3) = -18$

$\therefore \lim_{x\to -3^-} f(x) = \lim_{x\to -3^+} f(x) = f(-3)$

$\therefore$ f is continuous at $x = -3$

At $x = 5$

$f(5) = 5(5) - 3 = 22$

L.H.L. = $\lim_{x\to 5^-} f(x) = \lim_{h\to 0} [5(5-h)-3]$

$= \lim_{h\to 0} 25-5h-3 = 22$

R.H.L. = $\lim_{x\to 5^+} f(x) = \lim_{h\to 0} [5(5+h)-3]$

$= \lim_{h\to 0} 25+5h-3 = 22$

$\therefore \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$

$\therefore$ f is continuous at $x = 5$.

2. We have, $f(x) = 2x^2 - 1$

R.H.L. = $\lim_{x\to 3^+} f(x) = \lim_{h\to 0} 2(3+h)^2 - 1$

$= \lim_{h\to 0} 2(9+6h+h^2) - 1$

$= \lim_{h\to 0} (18+12h+2h^2) - 1$

$= \lim_{h\to 0} (17+12h+2h^2) = 17$

L.H.L. = $\lim_{x\to 3^-} f(x) = \lim_{h\to 0} 2(3-h)^2 - 1$

$= \lim_{h\to 0} 2(9-6h+h^2) - 1$

$= \lim_{h\to 0} (18-12h+2h^2) - 1$

$= \lim_{h\to 0} 2h^2 - 12h + 17 = 17$

Also, $f(3) = 2(3)^2 - 1 = 17$

$\therefore \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3)$

Hence, the given function $f(x) = 2x^2 - 1$ is continuous at $x = 3$.

3. (a) We have, $f(x) = x - 5$

Let a be a real number, then

$\lim_{x\to a^+} f(x) = \lim_{h\to 0} (a+h) - 5 = a - 5$

$\lim_{x\to a^-} f(x) = \lim_{h\to 0} (a-h) - 5 = a - 5$

Also $f(a) = a - 5$

$\because \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

Hence, the given function $f(x) = (x - 5)$ is continuous.

(b) We have, $f(x) = \frac{1}{x-5}$

Let a be a real number, then

$\lim_{x\to a^+} f(x) = \lim_{h\to 0} \frac{1}{a+h-5} = \frac{1}{a-5}$

$\lim_{x\to a^-} f(x) = \lim_{h\to 0} \frac{1}{a-h-5} = \frac{1}{a-5}$

$f(a) = \frac{1}{a-5}$ $\because \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

Hence, the given function $f(x) = \frac{1}{x-5}$ is continuous at all points except at $x = 5$.

(c) We have, $f(x) = \frac{x^2-25}{x+5} = \frac{(x+5)(x-5)}{(x+5)} = x - 5$

$\therefore$ $f(x) = x - 5$

Let a be a real number, then

$\lim_{x\to a^+} f(x) = \lim_{h\to 0} (a+h) - 5 = a - 5$ and

$\lim_{x\to a^-} f(x) = \lim_{h\to 0} (a-h) - 5 = a - 5$

Also, $f(a) = a - 5$

$\because \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

Hence, the given function $f(x) = x - 5$ is continuous at every point of its domain.

(d) We have, $f(x) = |x - 5|$

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} |a+h-5| = |a-5|$$

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} |a-h-5| = |a-5|$$

$f(a) = |a - 5|$

$\therefore \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

Hence the given function $f(x) = |x - 5|$ is continuous.

4. Given, $f(x) = x^n$, $n \in N$

So, $f(x)$ is a polynomial function and domain of $f = R$

$$\lim_{x\to n} f(x) = \lim_{x\to n} x^n = n^n = f(n)$$

$\Rightarrow$ f is continuous at $n \in N$.

5. (i) At $x = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x) = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x) = 0$$

Also $f(0) = 0$

Thus, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

(ii) At $x = 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x) = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 5 = 5$$

$\therefore \lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$

Hence, f is discontinuous at $x = 1$.

(iii) At $x = 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 5 = 5 \text{ and } \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 5 = 5$$

Also, $f(2) = 5$

Thus, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\therefore$ $f(x)$ is continuous at $x = 2$.

6. For $x < 2$, the function $f(x) = 2x + 3$ is polynomial hence continuous.

For $x > 2$, the function $f(x) = 2x - 3$ is polynomial, hence continuous.

For continuity at $x = 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+3) = \lim_{h\to 0} 2(2-h)+3$$

$$= \lim_{h\to 0} (4-2h+3) = \lim_{h\to 0} (7-2h) = 7$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2x-3) = \lim_{h\to 0} [2(2+h)-3]$$

$$= \lim_{h\to 0} (4+2h-3) = \lim_{h\to 0} (1+2h) = 1$$

Thus, $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

$\therefore$ $f(x)$ is not continuous at $x = 2$.

7. At $x = -3$

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} |x|+3 = \lim_{h\to 0} (|-3-h|+3)$$

$= |-3 - 0| + 3 = 3 + 3 = 6$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} (-2x) = \lim_{h\to 0} (-2(-3+h))$$

$= -2(-3 + 0) = 6$

$f(-3) = |-3| + 3 = 3 + 3 = 6.$

Thus, $\lim_{x\to -3^-} f(x) = \lim_{x\to -3^+} f(x) = f(-3)$

$\therefore$ f is continuous at $x = -3$.

At $x = 3$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (-2x) = \lim_{h\to 0} (-2(3-h))$$

$= -2(3 - 0) = -6$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (6x+2) = \lim_{h\to 0} (6(3+h)+2)$$

$= 6(3 + 0) + 2 = 20$

Thus, $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

$\therefore$ f is discontinuous at $x = 3$.

8. At $x = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} (1) = 1$$

Thus, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\Rightarrow$ f is discontinuous at $x = 0$

9. At $x = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{(-x)} \quad [\because x < 0]$$

$$= \lim_{x\to 0^-} (-1) = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (-1) = -1$$

Also $f(0) = -1$

Thus, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\Rightarrow$ f is continuous at $x = 0$

10. We observe that f is continuous at all real numbers $x < 1$ and $x > 1$.

Now, continuity at $x = 1$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+1) = \lim_{h\to 0} (1+h)+1 = 2$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x^2+1) = \lim_{h\to0}(1-h)^2+1$$
$$= \lim_{h\to0}(1-2h+h^2)+1 = 2$$

Also $f(1) = 2$

$\therefore \lim_{x\to1^+} f(x) = \lim_{x\to1^-} f(x) = f(1)$

$\therefore$ $f(x)$ is continuous at $x = 1$ and all points.

11. We observe that f is continuous at all real numbers $x < 2$ and $x > 2$.

Now, continuity at $x = 2$

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+}(x^2+1) = \lim_{h\to0}(2+h)^2+1$$
$$= \lim_{h\to0}(4+4h+h^2)+1 = 5$$
$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-}(x^3-3) = \lim_{h\to0}(2-h)^3-3$$
$$= \lim_{h\to0}(8-12h+6h^2-h^3)-3 = 5$$

Also $f(2) = 8 - 3 = 5$

$\therefore \lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x) = f(2)$

$\therefore$ $f(x)$ is continuous at $x = 2$ and all points.

12. $f(x) = \begin{cases} x^{10}-1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$

At $x = 1$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x^{10}-1)$$
$$= \lim_{h\to0}((1-h)^{10}-1) = 0$$
$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} x^2$$
$$= \lim_{h\to0}(1+h)^2$$
$$= \lim_{h\to0}(1+h^2+2h) = 1$$

L.H.L. $\ne$ R.H.L.

$\Rightarrow$ f is discontinuous at $x = 1$.

13. We observe that f is continuous at all real numbers $x < 1$ and $x > 1$.

Now, continuity at $x = 1$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(x-5) = \lim_{h\to0}(1+h-5)$$
$$= \lim_{h\to0}(h-4) = -4$$
$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x+5) = \lim_{h\to0}(1-h+5)$$
$$= \lim_{h\to0}(6-h) = 6$$

Thus, $\lim_{x\to1^+} f(x) \ne \lim_{x\to1^-} f(x)$

$\therefore$ $f(x)$ is not continuous at $x = 1$.

14. At $x = 1$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} 3 = 3$ and

$\text{Lim}_{x\to1^+} f(x) = \text{Lim}_{x\to1^+} 4 = 4$

L.H.L. $\ne$ R.H.L. at $x = 1$

At $x = 3$

$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} 4 = 4$ and

$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} 5 = 5$

L.H.L. $\ne$ R.H.L. at $x = 3$.

$\therefore$ Function is not continuous at $x = 1$ and 3.

15. At $x = 0$

$$\text{L.H.L.} = \lim_{x\to0^-} f(x) = \lim_{x\to0^-} 2x = \lim_{h\to0} 2(0-h) = 0$$
$$\text{R.H.L.} = \lim_{x\to0^+} f(x) = \lim_{x\to0^+} 0 = 0$$

Also $f(0) = 0$

$\therefore$ L.H.L. = R.H.L. = $f(x)$

So, $f(x)$ is continuous at $x = 0$

At $x = 1$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} 0 = 0$$

and $\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} 4x = \lim_{h\to0} 4(1+h)$

$$= \lim_{h\to0} 4+4h = 4$$

$\therefore$ L.H.L. $\ne$ R.H.L.

So, $f(x)$ is discontinuous at $x = 1$.

16. We have, $f(-1) = -2$

$$\text{R.H.L.} = \lim_{x\to-1^+} f(x) = \lim_{h\to0} 2(-1+h) = -2$$
$$\text{L.H.L.} = \lim_{x\to-1^-} f(x) = \lim_{h\to0}(-2) = -2$$

Also $f(-1) = -2$

$\therefore \lim_{x\to-1^-} f(x) = \lim_{x\to-1^+} f(x) = f(-1)$

Hence, $f(x)$ is continuous at $x = -1$.

At $x = 1$

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+}(2) = 2$$
$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{h\to0} 2(1-h) = 2$$

Also $f(1) = 2(1) = 2$

$\therefore \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

17. At $x = 3$

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^-}(ax+1) = \lim_{h\to0}(a(3-h)+1)$$
$$= \lim_{h\to0}(3a-ah+1) = 3a+1$$

$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+}(bx+3) = \lim_{h\to0}(b(3+h)+3)$$

$$= \lim_{h\to0}(3b+bh+3) = 3b+3$$

Also $f(3) = 3a + 1$

Thus, $\lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x) = f(3)$

$\Rightarrow$ $3b + 3 = 3a + 1$ $\Rightarrow$ $2 = 3(a - b)$

$\Rightarrow$ $a - b = \frac{2}{3}$ $\Rightarrow a = b + \frac{2}{3}$

18. Since $f(x)$ is continuous at $x = 0$

(i) L.H.L. = $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \lambda(x^2 - 2x)$

$= \lambda(0 - 0) = 0$

R.H.L. = $\lim_{x\to0^+} f(x) = \lim_{x\to0^+}(4x+1) = 4(0)+1 = 1$

Also, $f(0) = 0$

L.H.L. ≠ R.H.L.

For no value of λ, $f(x)$ is continuous at $x = 0$.

(ii) At $x = 1$

$\lim_{x\to1} f(x) = \lim_{x\to1}(4x+1) = f(1)$ for any value of λ.

Hence, f is continuous at $x = 1$ for all values of λ.

19. Let $n \in I$.

Then $\lim_{x\to n^-}[x] = n-1$

[$\because [x] = n - 1\ \forall\ x \in [n - 1, n]$]

and $g(n) = n - n = 0$. [$\because [n] = n$ because $n \in I$]

Now,

$$\lim_{x\to n^-} g(x) = \lim_{x\to n^-}(x-[x]) = \lim_{x\to n^-} x - \lim_{x\to n^-}[x]$$

$$= n - (n - 1) = 1$$

and $\lim_{x\to n^+} g(x) = \lim_{x\to n^+}(x-[x])$

$$= \lim_{x\to n^+} x - \lim_{x\to n^+}[x] = n - n = 0$$

Thus $\lim_{x\to n^-} g(x) \neq \lim_{x\to n^+} g(x)$.

Hence 'g' is discontinuous at all integral points.

20. At $x = \pi$

$$\lim_{x\to\pi^+} f(x) = \lim_{h\to0}(\pi+h)^2 - \sin(\pi+h) + 5$$

$$= \lim_{h\to0}\left[(\pi^2+h^2+2\pi h)+(\sin h+5)\right] = [\pi^2 + 5]$$

$$\Rightarrow \lim_{x\to\pi^-} f(x) = \lim_{h\to0}(\pi-h)^2 - \sin(\pi-h) + 5$$

$$= \lim_{h\to0}(\pi^2+h^2-2\pi h) - \sin h + 5 = \pi^2 + 5$$

$f(\pi) = \pi^2 + 5$

R.H.L. = L.H.L. = $f(\pi)$

$\therefore$ Function is continuous at $x = \pi$.

21. (a) Let a be an arbitrary real number then

$f(a) = \sin a + \cos a$

$$\lim_{x\to a^+} f(x) = \lim_{h\to0}[\sin(a+h)+\cos(a+h)]$$

$$= \lim_{h\to0}\{(\sin a\cos h + \cos a\sin h) + (\cos a\cosh - \sin a\sinh)\}$$

$= \sin a\cos0 + \cos a\sin0 + \cos a\cos0 - \sin a\sin0$

$= \sin a(1) + \cos a(0) + \cos a(1) - \sin a(0)$

$= \sin a + \cos a$

$$\lim_{x\to a^-} f(x) = \lim_{h\to0}[\sin(a-h)+\cos(a-h)]$$

$$= \lim_{h\to0}[(\sin a\cosh - \cos a\sin h) + (\cos a\cosh + \sin a\sinh)]$$

$= \sin a + \cos a$

$\therefore \lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$

$\Rightarrow$ $f(x)$ is continuous at $x = a$.

$\therefore$ $f(x) = \sin x + \cos x$ is everywhere continuous.

(b) Let a be an arbitrary real number then

$f(a) = \sin a - \cos a$

$$\lim_{x\to a^+} f(x) = \lim_{h\to0}\sin(a+h) - \cos(a+h)$$

$$= \lim_{h\to0}\{(\sin a\cos h + \cos a\sin h) - (\cos a\cosh - \sin a\sinh)$$

$= \sin a\cos0 + \cos a\sin0 - \cos a\cos0 + \sin a\sin0$

$= \sin a(1) + \cos a(0) - \cos a(1) + \sin a(0)$

$= \sin a - \cos a$

$$\lim_{x\to a^-} f(x) = \lim_{h\to0}[(\sin(a-h)-\cos(a-h)]$$

$$= \lim_{h\to0}[(\sin a\cosh - \cos a\sin h) - (\cos a\cosh + \sin a\sinh)]$$

$= \sin a - \cos a$.

$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

$\Rightarrow$ $f(x)$ is continuous at $x = a$.

So, $f(x) = \sin x - \cos x$ is everywhere continuous.

(c) Let a be an arbitrary real number then

$f(a) = \sin a\cos a$

$$\lim_{x\to a^+} f(x) = \lim_{h\to0}[\sin(a+h)\cos(a+h)]$$

$$= \lim_{h\to0}[(\sin a\cosh + \cos a\sin h)(\cos a\cosh - \sin a\sinh)]$$

$= \sin a\cos a$.

$$\lim_{x\to a^-} f(x) = \lim_{h\to0}[\sin(a-h)\cos(a-h)]$$

$$= \lim_{h\to0}[(\sin a\cosh - \cos a\sinh)(\cos a\cosh + \sin a\sinh)]$$

$= \sin a\cos a$

$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

$\Rightarrow$ $f(x)$ is continuous at $x = a$.

So, $f(x) = \sin x \cdot \cos x$ is everywhere continuous.

22. (a) $f(x) = \cos x$. Clearly, domain of $f = R$
Let a be an arbitrary real number, then
$f(a) = \cos a$

Now, $\lim_{x \to a^-} f(x) = \lim_{h \to 0} \cos(a - h)$

$= \lim_{h \to 0} (\cos a \cosh + \sin a \sinh) = \cos a$

$\lim_{x \to a^+} f(x) = \lim_{h \to 0} \cos(a + h)$

$= \lim_{h \to 0} (\cos a \cosh - \sin a \sinh) = \cos a$

$\therefore \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

$\therefore$ $f(x) = \cos x$ is continuous at a for all $a \in R$
Hence $\cos x$ is continuous.

(b) $f(x) = \operatorname{cosec} x \Rightarrow f(x) = \frac{1}{\sin x}$
and domain of $f = R - \{n\pi\}, n \in Z$

Also, $f(a) = \frac{1}{\sin a}$

$\lim_{x \to a^+} \frac{1}{\sin x} = \lim_{h \to 0} \frac{1}{\sin(a + h)}$

$= \lim_{h \to 0} \frac{1}{\sin a \cos h + \cos a \sin h}$

$= \frac{1}{\sin a\,(1) + \cos a\,(0)} = \frac{1}{\sin a + 0} = \frac{1}{\sin a}$

$\lim_{x \to a^-} f(x) = \lim_{h \to 0} \frac{1}{\sin(a - h)}$

$= \lim_{h \to 0} \frac{1}{\sin a \cos h - \cos a \sin h} = \frac{1}{\sin a}$

Thus cosec x is continuous except for $x = n\pi, n \in Z$.

(c) $f(x) = \sec x \Rightarrow f(x) = \frac{1}{\cos x}$

Clearly, Domain of $f = R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

Also, $f(a) = \frac{1}{\cos a}$

$\lim_{x \to a^+} f(x) = \lim_{h \to 0} \frac{1}{\cos(a + h)}$

$= \lim_{h \to 0} \frac{1}{\cos a \cos h - \sin a \sin h}$

$= \frac{1}{\cos a \cos 0 - \sin a \sin 0}$

$= \frac{1}{\cos a(1) - \sin a(0)} = \frac{1}{\cos a}$

$\therefore \lim_{x \to a^-} f(a) = \lim_{h \to 0} \frac{1}{\cos(a - h)}$

$= \lim_{h \to 0} \frac{1}{\cos a \cos h + \sin a \sin h} = \frac{1}{\cos a}$

$\therefore \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

Thus sec x is continuous except for

$x = (2n + 1)\frac{\pi}{2}, n \in Z$

(d) $f(x) = \cot x$

$f(x) = \frac{1}{\tan x}$ and Domain of $R - \{n\pi\}, n \in I$

$\lim_{x \to a^+} f(x) = \lim_{h \to 0} \frac{1}{\tan(a + h)}$

$\lim_{h \to 0} \frac{1}{\frac{\tan a + \tan h}{1 - \tan a \tan h}} = \frac{1}{\frac{\tan a + 0}{1 - \tan a \tan 0}} = \frac{1}{\frac{\tan a}{1 - 0}} = \frac{1}{\tan a}$

$\lim_{x \to a^-} f(x) = \lim_{h \to 0} \frac{1}{\tan(a - h)}$

$= \lim_{h \to 0} \frac{1}{\frac{\tan a - \tan h}{1 + \tan a \tan h}} = \frac{1}{\tan a}$

Thus, cot x is continuous except for $x = n\pi, n \in I$.

23. At $x = 0, f(0) = 1$

L.H.L. $= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} \frac{\sin(-h)}{-h} = 1$

R.H.L. $= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} (h + 1) = 0 + 1 = 1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\therefore$ f is continuous at $x = 0$

When $x < 0$, $\sin x$ and x both are continuous.

$\therefore$ $\frac{\sin x}{x}$ is also continuous.

When $x > 0$, $f(x) = x + 1$ is a polynomial.
$\therefore$ f is continuous.
$\Rightarrow$ f is not discontinuous at any point.

24. We have $f(0) = 0$

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} (0 - h)^2 \sin\frac{1}{-h} = \lim_{h \to 0}\left(-h^2 \sin\frac{1}{h}\right)$

But $\sin\frac{1}{h} \in [-1, 1] \Rightarrow h^2 \sin\frac{1}{h} \to 0$ as $h \to 0$.

$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} (0 + h)^2 \sin\frac{1}{h} = \lim_{h \to 0}\left(h^2 \sin\frac{1}{h}\right) = 0$

$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\Rightarrow$ f is continuous at $x = 0$.

25. We have

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} \sin(0 - h) - \cos(0 - h)$

$\lim_{h \to 0} (-\sin h - \cos h) = -(0) - 1 = -1$

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} [\sin(0+h) - \cos(0+h)]$

$= \lim_{h\to 0} (\sin h - \cos h) = 0 - 1$

Also, $f(0) = -1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\therefore$ f is continuous at $x = 0$.

At $x < 0$, $f(x) = \sin x - \cos x$ is continuous

At $x > 0$, $f(x) = \sin x - \cos x$ is also continuous

$\therefore$ $f(x)$ is continuous at all $x \in R$.

26. Since the function is continuous, then

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} f(x) = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi - 2\left(\frac{\pi}{2}-h\right)}$$

$$= \lim_{h\to 0} \frac{k\sin h}{\pi - \pi + 2h}$$

$$= \lim_{h\to 0} \frac{k\sin h}{2h} = \frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h} = \frac{k}{2}$$

$$\lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x) = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)} = \lim_{h\to 0} \frac{-k\sin h}{-2h}$$

$$= \frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h} = \frac{k}{2}$$

Also $f\left(\frac{\pi}{2}\right) = 3$

For continuity at $x = \frac{\pi}{2}$,

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} f(x) = \lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x) = f\left(\frac{\pi}{2}\right)$$

$\Rightarrow \frac{k}{2} = 3 \Rightarrow k = 6$

27. We have, $f(2) = 4k$.

$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} 3 = 3$

$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} k(2-h)^2 = 4k$

For continuity at $x = 2$,

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$4k = 3 \Rightarrow k = \frac{3}{4}$.

28. $\lim_{x\to \pi^+} f(x) = \lim_{h\to 0} f(\pi+h)$

$= \lim_{h\to 0} \cos(\pi+h) = \lim_{h\to 0} (-\cos h) = -\cos(0) = -1$

$\lim_{x\to \pi^-} f(x) = \lim_{h\to 0} k(\pi-h)+1 = k\pi+1$

$f(\pi) = k\pi + 1$

Since, the given function is continuous at $x = \pi$

$\therefore \lim_{x\to \pi^+} f(x) = \lim_{x\to \pi^-} f(x) = f(\pi)$

$k\pi + 1 = -1 \Rightarrow k\pi = -2$

$\Rightarrow k = \frac{-2}{\pi}$.

29. $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} (kx+1)$

$= \lim_{h\to 0} (k(5-h)+1) = k(5-0) + 1 = 5k + 1$

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} (3x-5) = \lim_{h\to 0} (3(5+h)-5)$

$= 3(5+0) - 5 = 10$

$f(5) = 5k + 1$

For continuity at $x = 5$,

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$

$5k + 1 = 10 \Rightarrow 5k = 9 \Rightarrow k = \frac{9}{5}$.

30. Since f is continuous at all x,

$\therefore$ f is continuous at $x = 2, 10$.

At $x = 2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (5) = 5$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax+b)$

$= \lim_{h\to 0} (a(2+h)+b) = a(2+0)+b = 2a+b$

Also, $f(2) = 5$.

For continuity, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\Rightarrow 2a + b = 5$ ...(i)

At $x = 10$

$\lim_{x\to 10^-} f(x) = \lim_{x\to 10^-} (ax+b)$

$= \lim_{h\to 0} (a(10-h)+b) = a(10-0)+b = 10a+b$

$\lim_{h\to 10^+} f(x) = \lim_{h\to 10^+} (21) = 21$

Also, $f(10) = 21$.

For continuity, $\lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$

$\Rightarrow 10a + b = 21$ ...(ii)

Subtracting (i) from (ii), we get

$8a = 16 \Rightarrow a = 2$.

Putting $a = 2$ in (i), we get

$2(2) + b = 5 \Rightarrow b = 5 - 4 = 1$.

Hence $a = 2$, $b = 1$.

31. Let $f(x) = \cos(x^2)$. Domain of $f = R$.
Let a be any arbitrary real number.

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} \cos(a-h)^2 = \cos a^2$$

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} \cos(a+h)^2 = \cos a^2$$

Also $f(a) = \cos a^2$

Thus, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)\ \forall\ a \in R.$

$\therefore$ $f(x) = \cos(x^2)$ is continuous at $a\ \forall\ a \in R$.

32. We know that cosine function is every where continuous and also modulus function is continuous. Therefore $|\cos x|$ is everywhere continuous.

33. Let $f(x) = |x|$ and $g(x) = \sin x$. Then
$(gof)(x) = g[f(x)] = g(|x|) = \sin|x|$
Now, f and g being continuous, it follows that their composite (gof) is also continuous.

34. We have

$$f(x) = \begin{cases} -(x)-[-(x+1)], & \text{if} \quad x<-1 \\ -(x)-(x+1), & \text{if} \quad -1\le x<0 \\ (x)-(x+1), & \text{if} \quad x\ge 0 \end{cases}$$

$$f(x) = \begin{cases} 1, & \text{if} \quad x<-1 \\ -2x-1, & \text{if} \quad -1\le x<0 \\ -1, & \text{if} \quad x\ge 0 \end{cases}$$

At $x = -1$

$$\lim_{x\to -1^-} f(x) = 1$$

$$\lim_{x\to -1^+} f(x) = \lim_{h\to 0}(-2(-1+h)-1) = 1$$

$f(-1) = -2(-1) - 1 = 1$

Thus, $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$

$\Rightarrow$ f is continuous at $x = -1$

At $x = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-2x-1)$$

$$= \lim_{h\to 0}(-2(-h)-1) = -1$$

$$\lim_{x\to 0^+} f(x) = -1$$

Also, $f(0) = -1$

Thus, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0).$

f is continuous at $x = 0$.
Also, f being a constant, is continuous when $x < -1$ or when $x > 0$.
$\therefore$ f is continuous for all $x \in R$
Hence, there is no point of discontinuity.

EXERCISE - 5.2

1. Let $y = \sin(x^2 + 5)$

$$\frac{dy}{dx} = \frac{d}{dx}\sin(x^2+5) = \cos(x^2+5)\frac{d}{dx}(x^2+5)$$

$= \cos(x^2 + 5)\,(2x + 0) = 2x\cos(x^2 + 5)$.

2. Let $y = \cos(\sin x)$

$$\frac{dy}{dx} = \frac{d}{dx}\cos(\sin x) = -\sin(\sin x)\frac{d}{dx}\sin x$$

$= -\sin(\sin x)\cos x$.

3. Let $y = \sin(ax + b)$

$$\frac{dy}{dx} = \frac{d}{dx}\sin(ax+b) = \cos(ax+b)\frac{d}{dx}(ax+b)$$

$= \cos(ax + b)\,(a + 0) = a\cos(ax + b)$.

4. Let $y = \sec\{\tan(\sqrt{x})\}$

$$\frac{dy}{dx} = \frac{d}{dx}\sec(\tan\sqrt{x})$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\frac{d}{dx}\tan\sqrt{x}$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}\frac{d}{dx}(\sqrt{x})$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}\cdot\frac{1}{2}x^{\frac{1}{2}-1}$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

5. Let $y = \dfrac{\sin(ax+b)}{\cos(cx+d)}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin(ax+b)}{\cos(cx+d)}\right)$$

$$= \frac{\cos(cx+d)\frac{d}{dx}\sin(ax+b) - \sin(ax+b)\frac{d}{dx}\cos(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{a\cos(cx+d)\cos(ax+b) + c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$$

$= a\cos(ax + b)\sec(cx + d) + c\sin(ax + b)\tan(cx + d)\sec(cx + d)$.

6. Let $y = \cos x^3\cdot\sin^2(x^5)$

$$\frac{dy}{dx} = \frac{d}{dx}[\cos x^3\cdot\sin^2(x^5)]$$

$$= \cos x^3\frac{d}{dx}\sin^2(x^5) + \sin^2(x^5)\frac{d}{dx}\cos x^3$$

$$= \cos x^3\cdot 2\sin(x^5)\frac{d}{dx}\sin(x^5) + \sin^2(x^5)(-\sin x^3)\frac{d}{dx}(x^3)$$

$= \cos x^3 \cdot 2\sin(x^5)\cos(x^5)\frac{d}{dx}(x^5) + \sin^2(x^5)(-\sin x^3)(3x^2)$

$= \cos x^3 \cdot 2\sin(x^5)\cos(x^5)(5x^4) - \sin^2(x^5)\sin x^3 \cdot (3x^2)$

$= 10x^4 \cos x^3 \sin(x^5)\cos(x^5) - 3x^2 \sin^2(x^5)\sin x^3$.

7. Let $y = 2\sqrt{\cot(x^2)}$

$$\frac{dy}{dx} = 2\frac{d}{dx}\sqrt{\cot(x^2)} = 2\cdot\frac{1}{2}\{\cot(x^2)\}^{\frac{-1}{2}}\cdot\frac{d}{dx}\cot(x^2)$$

$$= \frac{1}{\sqrt{\cot(x^2)}}\cdot\{-\operatorname{cosec}^2(x^2)\}\frac{d}{dx}(x^2)$$

$$= \frac{1}{\sqrt{\cot(x^2)}}\cdot\{-\operatorname{cosec}^2(x^2)\}(2x) = \frac{-2x\operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}$$

$$= \frac{-2x}{\sin^2 x^2}\times\frac{1}{\frac{\sqrt{\cos x^2}}{\sqrt{\sin x^2}}} = \frac{-2x}{\sin x^2\sqrt{\sin x^2}\sqrt{\cos x^2}}$$

$$= \frac{-2x\sqrt{2}}{\sin x^2\sqrt{2\sin x^2\cos x^2}} = \frac{-2\sqrt{2}x}{\sin x^2\sqrt{\sin 2x^2}}.$$

8. Let $y = \cos\left(\sqrt{x}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}\cos\left(\sqrt{x}\right) = -\sin\sqrt{x}\cdot\frac{d}{dx}(\sqrt{x})$$

$$= -\sin\sqrt{x}\cdot\frac{1}{2}(x)^{\frac{-1}{2}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

9. We have

$f(x) = |x-1|$

$f(1) = |1-1| = 0$

$$Rf'(1) = \lim_{h\to0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0}\frac{|1+h-1|-0}{h}$$

$$\lim_{h\to0}\frac{|h|}{h} = \lim_{h\to0}\frac{h}{h} = 1 \text{ and}$$

$$Lf'(1) = \lim_{h\to0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to0}\frac{|1-h-1|-0}{-h}$$

$$\lim_{h\to0}\frac{|-h|}{-h} = \lim_{h\to0}\frac{h}{-h} = -1$$

Thus $Rf'(1) \ne Lf'(1)$

This shows that $f(x)$ is not differentiable at $x = 1$.

10. At $x = 1$,

$$Rf'(1) = \lim_{h\to0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to0}\frac{[1+h]-[1]}{h} = 0 \qquad [\because [1+h]=1 \text{ and } [1]=1]$$

and $$Lf'(1) = \lim_{h\to0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to0}\frac{[1-h]-[1]}{-h} = \infty \qquad [\because [1-h]=0 \text{ and } [1]=1]$$

Thus, $Rf'(1) \ne Lf'(1)$

Hence $f(x) = [x]$ is not differentiable at $x = 1$.

At $x = 2$,

$$Rf'(2) = \lim_{h\to0}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to0}\frac{[2+h]-[2]}{h} \qquad \begin{bmatrix}\because [2+h]=2\\ [2]=2\end{bmatrix}$$

$$= \frac{2-2}{h} = 0$$

$$Lf'(2) = \lim_{h\to0}\frac{f(2-h)-f(2)}{-h} \qquad [\because [2-h]=1, [2]=2]$$

$$= \lim_{h\to0}\frac{[2-h]-[2]}{-h}$$

$$= \frac{1-2}{0} = \infty$$

$\therefore$ $Rf'(2) \ne Lf'(2)$

Hence, $f(x)$ is not differentiable at $x = 2$.

EXERCISE - 5.3

1. We are given that $2x + 3y = \sin x$ ...(i)

Differentiating both sides of (i) w.r.t. x, we get

$$2 + 3\frac{dy}{dx} = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

2. We are given that $2x + 3y = \sin y$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$2 + 3\frac{dy}{dx} = \cos y\frac{dy}{dx}$$

$$\Rightarrow \cos y\frac{dy}{dx} - 3\frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

3. We are given that $ax + by^2 = \cos y$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$a + b\left[2y\frac{dy}{dx}\right] = -\sin y\frac{dy}{dx}$$

$$\Rightarrow a + 2by\frac{dy}{dx} + \sin y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}[2by + \sin y] = -a \Rightarrow \frac{dy}{dx} = \frac{-a}{2by + \sin y}$$

4. We are given that $xy + y^2 = \tan x + y$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x - y$$

$$\Rightarrow \frac{dy}{dx}[x + 2y - 1] = \sec^2 x - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}$$

5. We are given that $x^2 + xy + y^2 = 100$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}[x + 2y] = -(2x + y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(2x + y)}{(x + 2y)}$$

6. We are given that

$x^3 + x^2y + xy^2 + y^3 = 81$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$3x^2 + x^2\frac{dy}{dx} + y(2x) + y^2 + x\left(2y\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}[x^2 + 2xy + 3y^2] = -(3x^2 + 2xy + y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}$$

7. We are given that

$\sin^2 y + \cos xy = k$ ...(i)

Differentiating both sides w.r.t. x, we get

$$2\sin y\frac{d}{dx}(\sin y) + (-\sin xy)\frac{d}{dx}(xy) = 0$$

$$\Rightarrow 2\sin y\cos y\frac{dy}{dx} + (-\sin xy)\left[x\frac{dy}{dx} + y\right] = 0$$

$$\Rightarrow 2\sin y\cos y\frac{dy}{dx} - x\sin xy\frac{dy}{dx} - y\sin xy = 0$$

$$\Rightarrow \frac{dy}{dx}[2\sin y\cos y - x\sin xy] = y\sin xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

8. We are given that $\sin^2 x + \cos^2 y = 1$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$2\sin x\frac{d}{dx}(\sin x) + 2\cos y\frac{d}{dx}(\cos y) = 0$$

$$\Rightarrow 2\sin x\cos x + 2\cos y(-\sin y)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

9. We are given that $y = \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$

Putting $x = \tan\theta$, we get

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\tan^{-1}x$$

$$\therefore \frac{dy}{dx} = \frac{2}{1 + x^2}$$

10. We are given that $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

$\Rightarrow y = \tan^{-1}(\tan 3\theta)$

$\Rightarrow y = 3\theta \Rightarrow y = 3\tan^{-1}x$ $[\because x = \tan\theta \Rightarrow \theta = \tan^{-1}x]$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{1 + x^2}.$$

11. We are given that $y = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$, where $0 < x < 1$

Putting $x = \tan\theta$, we have

$$y = \cos^{-1}\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) \Rightarrow y = \cos^{-1}(\cos 2\theta)$$

$\Rightarrow y = 2\theta$

$$\Rightarrow y = 2\tan^{-1}x \Rightarrow \frac{dy}{dx} = \frac{2}{1 + x^2}$$

12. We are given that $y = \sin^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$

Putting $x = \tan\theta$, we get

$$y = \sin^{-1}\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) \Rightarrow y = \sin^{-1}(\cos 2\theta)$$

$$\Rightarrow y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\} \Rightarrow y = \frac{\pi}{2} - 2\theta$$

$$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1}x \Rightarrow \frac{dy}{dx} = 0 - \frac{2}{1 + x^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{1 + x^2}.$$

13. We are given that $y = \cos^{-1}\left(\frac{2x}{1 + x^2}\right)$

Putting $x = \tan\theta$, we get

$$y = \cos^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right) \Rightarrow y = \cos^{-1}(\sin 2\theta)$$

$$\Rightarrow y = \cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 2\theta\right)\right\} \Rightarrow y = \frac{\pi}{2} - 2\theta$$

$$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1}x \Rightarrow \frac{dy}{dx} = -\frac{2}{1 + x^2}.$$

14. We are given that $y = \sin^{-1}(2x\sqrt{1 - x^2})$

Putting $x = \sin\theta$, we get $y = \sin^{-1}[2\sin\theta\sqrt{1 - \sin^2\theta}]$

$\Rightarrow y = \sin^{-1}(\sin 2\theta) \Rightarrow y = 2\theta$

$y = 2\sin^{-1}x \Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

15. We are given that $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$

Putting $x = \cos\theta$, we get

$y = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right)$

$\Rightarrow y = \cos^{-1}(2\cos^2\theta - 1)$ $\left[\because \sec^{-1}\frac{1}{x} = \cos^{-1}x\right]$

$\Rightarrow y = \cos^{-1}(\cos 2\theta)$

$\Rightarrow y = 2\theta \Rightarrow y = 2\cos^{-1}x$

$\Rightarrow \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$

EXERCISE - 5.4

1. Let $y = \frac{e^x}{\sin x}$

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(\sin x)}{\sin^2 x}$

$= \frac{\sin x \cdot e^x - e^x \cdot \cos x}{\sin^2 x} = \frac{e^x(\sin x - \cos x)}{\sin^2 x}, x \neq n\pi, n \in Z$

2. Let $y = e^{\sin^{-1}x}$

$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin^{-1}x}) = e^{\sin^{-1}x}\frac{d}{dx}(\sin^{-1}x)$

$= e^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}.$

3. Let $y = e^{x^3}$

$\frac{dy}{dx} = \frac{d}{dx}e^{x^3} = e^{x^3}\frac{d}{dx}x^3 = e^{x^3} \cdot 3x^2 = 3e^{x^3} \cdot x^2$

4. Let $y = \sin(\tan^{-1} e^{-x})$

$\frac{dy}{dx} = \frac{d}{dx}(\sin(\tan^{-1}(e^{-x})))$

$= \cos(\tan^{-1} e^{-x})\frac{d}{dx}(\tan^{-1}(e^{-x}))$

$= \cos(\tan^{-1} e^{-x}) \cdot \frac{1}{(1+e^{-2x})} \cdot \frac{d}{dx}e^{-x}$

$= \cos(\tan^{-1} e^{-x}) \cdot \frac{-e^{-x}}{(1+e^{-2x})}$

5. Let $y = \log(\cos e^x)$

$\frac{dy}{dx} = \frac{d}{dx}\log(\cos e^x) = \frac{1}{\cos e^x}\frac{d}{dx}(\cos e^x)$

$= \frac{1}{\cos e^x} \cdot (-\sin e^x)\frac{d}{dx}(e^x) = -\tan e^x \cdot e^x$

$= -e^x \tan e^x$

6. Let $y = e^x + e^{x^2} + e^{x^3} + \ldots + e^{x^5}$

$\frac{dy}{dx} = \frac{d}{dx}(e^x + e^{x^2} + \ldots + e^{x^5})$

$= \frac{d}{dx}(e^x) + \frac{d}{dx}(e^{x^2}) + \frac{d}{dx}(e^{x^3}) + \frac{d}{dx}(e^{x^4}) + \frac{d}{dx}(e^{x^5})$

$= e^x + e^{x^2}(2x) + e^{x^3}(3x^2) + e^{x^4}(4x^3) + e^{x^5}(5x^4)$

$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$

7. Let $y = \sqrt{e^{\sqrt{x}}}$

$\frac{dy}{dx} = \frac{d}{dx}\sqrt{e^{\sqrt{x}}} = \frac{d}{dx}(e^{\sqrt{x}})^{1/2} = \frac{1}{2}(e^{\sqrt{x}})^{-\frac{1}{2}} \cdot \frac{d}{dx}e^{\sqrt{x}}$

$= \frac{1}{2} \cdot (e^{\sqrt{x}})^{-\frac{1}{2}} \cdot e^{\sqrt{x}}\frac{d}{dx}\sqrt{x} = \frac{1}{2}(e^{\sqrt{x}})^{-1/2} \cdot e^{\sqrt{x}} \cdot \frac{1}{2}(x)^{-1/2}$

$= \frac{e^{\sqrt{x}}}{4\sqrt{x \cdot e^{\sqrt{x}}}}, x > 0$

8. Let $y = \log(\log x), x > 1$

$\frac{dy}{dx} = \frac{d}{dx}\log(\log x) = \frac{1}{(\log x)} \cdot \frac{d}{dx}(\log x)$

$= \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x\log x}, x > 1$

9. Let $y = \frac{\cos x}{\log x}$

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{\log x}\right) = \frac{\log x \frac{d}{dx}\cos x - \cos x \frac{d}{dx}\log x}{(\log x)^2}$

$= \frac{\log x(-\sin x) - \cos x\left(\frac{1}{x}\right)}{(\log x)^2} = -\left(\frac{\sin x \log x + \frac{1}{x}\cos x}{(\log x)^2}\right)$

$= \frac{-(x\sin x\log x + \cos x)}{x(\log x)^2}, x > 0$

10. Let $y = \cos(\log x + e^x)$

$\frac{dy}{dx} = \frac{d}{dx}\cos(\log x + e^x)$

$= -\sin(\log x + e^x)\frac{d}{dx}(\log x + e^x)$

$= -\sin(\log x + e^x)\left[\frac{1}{x} + e^x\right]$

$= -\left(\frac{1}{x} + e^x\right)\sin(\log x + e^x), x > 0$

EXERCISE - 5.5

1. Let $y = \cos x \cdot \cos 2x \cdot \cos 3x$

Taking log on both sides, we get

$\Rightarrow \log y = \log(\cos x \cdot \cos 2x \cdot \cos 3x)$

Then, $\log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$ ...(i)

On differentiating (i) both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{\cos x}(-\sin x)+\frac{1}{\cos 2x}(-\sin 2x)(2)+\frac{1}{\cos 3x}(-\sin 3x)(3)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=-\tan x-2\tan 2x-3\tan 3x$$

$$\therefore \frac{dy}{dx}=-\cos x\cdot\cos 2x\cdot\cos 3x\quad(\tan x+2\tan 2x+3\tan 3x)$$

2. Let $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ ...(i)

Taking log on both sides of (i), we get

$$\log y=\frac{1}{2}[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)] \quad ...(ii)$$

Differentiating (ii) on both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{(x-1)}+\frac{1}{(x-2)}-\frac{1}{(x-3)}-\frac{1}{(x-4)}-\frac{1}{(x-5)}\right]$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{(x-1)}+\frac{1}{(x-2)}-\frac{1}{(x-3)}-\frac{1}{(x-4)}-\frac{1}{(x-5)}\right]$$

3. Let $y=(\log x)^{\cos x}$ ...(i)

Taking log on both sides of (i), we get

$\log y=\cos x\log(\log x)$ ...(ii)

On differentiating (ii) both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=\cos x\frac{d}{dx}\log(\log x)+\log(\log x)\frac{d}{dx}\cos x$$

$$=\cos x\cdot\frac{1}{\log x}\frac{1}{x}+\log(\log x)(-\sin x)$$

$$=\frac{\cos x}{x\log x}-\sin x\log(\log x)$$

$$\frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin x\log(\log x)\right]$$

4. Let $y=x^x-2^{\sin x}$

$\Rightarrow$ $y=u-v$, where $u=x^x$ and $v=2^{\sin x}$.

Differentiating both sides w.r.t. x, we get

$$\therefore \frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx} \quad ...(i)$$

Now, $u=x^x$

Taking log on both sides, we get

$\log u=x\log x$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=x\frac{1}{x}+\log x$$

$$\Rightarrow \frac{du}{dx}=x^x[1+\log x] \quad ...(ii)$$

and $v=2^{\sin x}$

Taking log on both sides, we get

$\log v=\sin x\log 2$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\log 2(\cos x)$$

$$\Rightarrow \frac{dv}{dx}=2^{\sin x}(\cos x\cdot\log 2) \quad ...(iii)$$

From (i), (ii) and (iii) we get

$$\frac{dy}{dx}=x^x[1+\log x]-2^{\sin x}(\cos x\cdot\log 2)$$

5. Let $y=(x+3)^2\cdot(x+4)^3\cdot(x+5)^4$

Taking log on both sides, we get

$\log y=\log[(x+3)^2(x+4)^3(x+5)^4]$ ...(i)

$\log y=2\log(x+3)+3\log(x+4)+4\log(x+5)$

Differentiating (i) w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=2\cdot\frac{1}{(x+3)}+3\cdot\frac{1}{(x+4)}+4\cdot\frac{1}{(x+5)}$$

$$\Rightarrow \frac{dy}{dx}=(x+3)^2\cdot(x+4)^3\cdot(x+5)^4\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right].$$

$$\Rightarrow \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$$

6. Let $y=\left(x+\frac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}=u+v,$

where $u=\left(x+\frac{1}{x}\right)^x$ and $v=x^{\left(1+\frac{1}{x}\right)}$

Differentiating the above w.r.t. x, we get

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad ...(i)$$

Now, $u=\left(x+\frac{1}{x}\right)^x$

Taking log on both sides, we get

$$\Rightarrow \log u=x\log\left(x+\frac{1}{x}\right) \quad ...(ii)$$

Differentiating (ii) w.r.t. x, we get

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=x\frac{d}{dx}\log\left(x+\frac{1}{x}\right)+\log\left(x+\frac{1}{x}\right)(1)$$

$$=\frac{x}{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right)+\log\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x}{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right)+\log\left(x+\frac{1}{x}\right)\right] \quad ...(iii)$$

Also, $v=x^{\left(1+\frac{1}{x}\right)}$

Taking log on both sides, we get

$$\Rightarrow \quad \log v = \left(1+\frac{1}{x}\right)\log x \quad \text{...(iv)}$$

Differentiating (iv) w.r.t. x, we get

$$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = \left(1+\frac{1}{x}\right)\frac{d}{dx}\log x + \log x\frac{d}{dx}\left(1+\frac{1}{x}\right)$$

$$= \left(1+\frac{1}{x}\right)\frac{1}{x} + \log x\left(-\frac{1}{x^2}\right)$$

$$\frac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left[\left(1+\frac{1}{x}\right)\frac{1}{x} + \log x\left(\frac{-1}{x^2}\right)\right] \quad \text{...(v)}$$

Substituting the value of (iii) and (v) in (i), we get

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x}{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right) + \log\left(x+\frac{1}{x}\right)\right]$$

$$+ x^{\left(1+\frac{1}{x}\right)}\left[\left(1+\frac{1}{x}\right)\frac{1}{x} + \log x\left(\frac{-1}{x^2}\right)\right]$$

7. Let $y = (\log x)^x + x^{\log x} = u + v$,

where $u = (\log x)^x$ and $v = x^{\log x}$

Differentiating $y = u + v$ w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

Now $u = (\log x)^x$

Taking log on both sides, we get

$\log u = x \log(\log x)$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = x\frac{d}{dx}\log(\log x) + \log(\log x)$$

$$= x\cdot\frac{1}{\log x}\cdot\frac{1}{x} + \log(\log x)$$

$$= \frac{1}{\log x} + \log(\log x)$$

$$\frac{du}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] \quad \text{...(iii)}$$

Also, $v = x^{\log x}$

Taking log on both sides, we get

$\log v = \log x \log x = (\log x)^2$ ...(iv)

Differentiating (iv) w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = 2\log x\cdot\frac{1}{x}$$

$$\frac{dv}{dx} = x^{\log x}\left[\frac{2\log x}{x}\right] \quad \text{...(v)}$$

$$\therefore \quad \frac{dy}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right]$$

$$+ x^{\log x}\left[\frac{2\log x}{x}\right] \quad \text{[From (i), (iii) and (v)]}$$

8. Let $y = (\sin x)^x + \sin^{-1}\sqrt{x} = u + v$,

where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

Differentiating $y = u + v$ w.r.t. x, we get

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

Now, $u = (\sin x)^x$

Taking log on both sides, we get

$\log u = x \log \sin x$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = x\frac{d}{dx}(\log \sin x) + \log \sin x$$

$$= x\cdot\frac{1}{\sin x}\cdot\cos x + \log \sin x$$

$= x\cot x + \log \sin x$

$$\frac{du}{dx} = (\sin x)^x\,[x\cot x + \log \sin x] \quad \text{...(iii)}$$

Also, $v = \sin^{-1}\sqrt{x}$ ...(iv)

Differentiating (iv) w.r.t. x, we get

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} \quad \text{...(v)}$$

$$\frac{dy}{dx} = (\sin x)^x\,[x\cdot\cot x + \log \sin x] + \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$$

[From (i), (iii) and (v)]

9. Let $y = x^{\sin x} + (\sin x)^{\cos x} = u + v$, where $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$.

Differentiating $y = u + v$ w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{.... (i)}$$

Now, $u = x^{\sin x}$

Taking log on both sides, we get

$\log u = \sin x \log x$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = \sin x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(\sin x)$$

$$= \sin x\cdot\frac{1}{x} + \log x\cos x$$

$$\frac{du}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x\cos x\right] \quad \text{...(iii)}$$

Also, $v = (\sin x)^{\cos x}$

Taking log on both sides, we get

$\log v = \cos x \log \sin x$ ...(iv)

Differentiating (iv) w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \cos x\frac{d}{dx}\log \sin x + \log \sin x\frac{d}{dx}\cos x$$

$$= \cos x\frac{1}{\sin x}\cos x + \log \sin x\,(-\sin x)$$

$= \cos x \cot x - \sin x \log \sin x$

$$\frac{dv}{dx} = (\sin x)^{\cos x} [\cos x \cot x - \sin x \log \sin x] \quad \text{...(v)}$$

Substituting the values of (iii) & (v) in (i), we get

$$\therefore \frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x \cos x\right]$$

$+ (\sin x)^{\cos x} [\cos x \cot x - \sin x \log \sin x]$

10. Let $y = x^{x\cos x} + \dfrac{x^2+1}{x^2-1} = u + v,$

where $u = x^{x\cos x}$ and $v = \dfrac{x^2+1}{x^2-1}$

Differentiating the above w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

Now, $u = x^{x\cos x}$

Taking log on both sides, we get

$\log u = x \cos x \log x$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x\cos x \frac{d}{dx}(\log x) + x \log x \frac{d}{dx}\cos x + \cos x \log x \frac{d}{dx}(x)$$

$$\frac{1}{u}\frac{du}{dx} = x\cos x \cdot \frac{1}{x} + x\log x(-\sin x) + \cos x \log x$$

$= \cos x - x \sin x \log x + \cos x \log x$

$$\frac{du}{dx} = x^{x\cos x}[\cos x - x\sin x \log x + \cos x \log x] \quad \text{...(iii)}$$

Also, $v = \dfrac{x^2+1}{x^2-1}$ ...(iv)

Differentiating (iv) w.r.t. x, we get

$$\Rightarrow \frac{dv}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$$

$$= \frac{2x[x^2-1-x^2-1]}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2} \quad \text{...(v)}$$

Substituting the values of (iii) & (v) in (i), we get

$$\therefore \frac{dy}{dx} = x^{x\cos x}[\cos x - x\sin x \log x + \cos x \log x] - \frac{4x}{(x^2-1)^2}$$

11. Let $y = (x \cos x)^x + (x \sin x)^{1/x} = u + v$, where $u = (x \cos x)^x$ and $, v = (x\sin x)^{1/x}$

On differentiating above w.r.t. x, we get

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

Now, $u = (x \cos x)^x$. Taking log on both sides, we get

$\log u = x \log (x \cos x)$ ...(ii)

Differentiating (ii) both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log(x\cos x) + \log(x\cos x)$$

$$= x\left[\frac{1}{x\cos x}\frac{d}{dx}(x\cos x)\right] + \log(x\cos x)$$

$$= x\left[\frac{1}{x\cos x}\left(x\frac{d}{dx}\cos x + \cos x\right)\right] + \log(x\cos x)$$

$$= x\left[\frac{1}{x\cos x}(x(-\sin x) + \cos x)\right] + \log(x\cos x)$$

$= \sec x (\cos x - x \sin x) + \log(x \cos x)$

$= 1 - x \tan x + \log(x \cos x)$

$$\frac{du}{dx} = (x\cos x)^x [1 - x\tan x + \log(x\cos x)] \quad \text{...(iii)}$$

Now, $v = (x \sin x)^{1/x}$

Taking log on both sides, we get

$$\log v = \frac{1}{x}\log(x\sin x) \quad \text{...(iv)}$$

Differentiating (iv) w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x}\frac{d}{dx}\log(x\sin x) + \log(x\sin x)\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\cdot\frac{1}{x\sin x}\cdot\frac{d}{dx}(x\sin x) + \log(x\sin x)\cdot\left(\frac{-1}{x^2}\right)$$

$$= \frac{1}{x^2\sin x}\left[x\frac{d}{dx}(\sin x) + \sin x\right] - \frac{\log(x\sin x)}{x^2}$$

$$= \frac{\cot x}{x} + \frac{1}{x^2} - \frac{\log(x\sin x)}{x^2}$$

$$= \frac{x\cot x + 1 - \log(x\sin x)}{x^2}$$

$$\frac{dv}{dx} = (x\sin x)^{1/x}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right] \quad \text{...(v)}$$

Substituting the values of (iii) and (v) in (i), we get

$$\frac{dy}{dx} = (x\cos x)^x[1 - x\tan x + \log(x\cos x)]$$

$$+ (x\sin x)^{1/x}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

12. $x^y + y^x = 1$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{d}{dx}(x^y) + \frac{d}{dx}(y^x) = 0 \quad \text{...(ii)}$$

Let $u = x^y$

Taking log on both sides, we get

$\log u = y \log x$ ...(iii)

Differentiating the above w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = y\cdot\frac{1}{x} + \log x\frac{dy}{dx}$$

$$\frac{du}{dx} = x^y\left[\frac{y}{x} + \log x\frac{dy}{dx}\right] \quad ...(iii)$$

Let $v = y^x \Rightarrow \log v = x \log y$ ...(iv)

Differentiating above w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = x\cdot\frac{1}{y}\frac{dy}{dx} + \log y$$

$$\Rightarrow \frac{dv}{dx} = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] \quad ...(v)$$

Substituting values of (iii) and (v) in (ii), we get

$$x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) + y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = 0$$

$$\Rightarrow (x^y \log x + xy^{x-1})\frac{dy}{dx} = -(y^x \log y + yx^{y-1})$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^x \log y + yx^{y-1}}{x^y \log x + xy^{x-1}}.$$

13. We are given that $y^x = x^y$

Taking log on both sides, we get

$x \log y = y \log x$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$x\cdot\frac{d}{dx}\log y + \log y\cdot 1 = y\frac{d}{dx}\log x + \log x\frac{dy}{dx}$$

$$\Rightarrow x\cdot\frac{1}{y}\frac{dy}{dx} + \log y = \frac{y}{x} + \log x\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[\log x - \frac{x}{y}\right] = \log y - \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(x\log y - y)}{x(y\log x - x)} = \frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$$

14. We have $(\cos x)^y = (\cos y)^x$

Taking log on both sides, we get

$\Rightarrow \quad y \log(\cos x) = x \log(\cos y)$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$y\frac{d}{dx}\log(\cos x) + \log(\cos x)\frac{dy}{dx}$$

$$= x\frac{d}{dx}(\log(\cos y)) + \log(\cos y)\cdot 1$$

$$\Rightarrow \quad y\frac{1}{\cos x}(-\sin x) + \log(\cos x)\frac{dy}{dx}$$

$$= x\cdot\frac{1}{\cos y}(-\sin y)\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \frac{dy}{dx}[\log(\cos x) + x\tan y] = \log(\cos y) + y\tan x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log(\cos y) + y\tan x}{\log(\cos x) + x\tan y}$$

15. We know that $xy = e^{(x-y)}$

Taking log on both sides, we get

$\log(xy) = \log e^{(x-y)}$ ...(i)

$\log x + \log y = x - y$

Differentiating (i) both sides w.r.t. x, we get

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = \left(1 - \frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx}\left(\frac{1}{y} + 1\right) = 1 - \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}.$$

16. Let $f(x) = y$

$y = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$

Taking log on both sides, we get

$\log y = \log[(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)]$

$\Rightarrow \log y = \log(1 + x) + \log(1 + x^2) + \log(1 + x^4)$
$+ \log(1 + x^8)$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{(1+x)} + \frac{1}{(1+x^2)}(2x) + \frac{1}{(1+x^4)}(4x^3) + \frac{1}{(1+x^8)}(8x^7)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{(1+x^4)} + \frac{8x^7}{(1+x^8)}\right]$$

i.e., $f'(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$

$$\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

$\Rightarrow f'(1) = (1 + 1)(1 + 1)(1 + 1)(1 + 1)$

$$\left[\frac{1}{1+1} + \frac{2(1)}{1+1} + \frac{4(1)}{1+1} + \frac{8(1)}{1+1}\right]$$

$$f'(1) = (2)(2)(2)(2)\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right]$$

$$f'(1) = 16\left\{\frac{1+2+4+8}{2}\right\} = \frac{16}{2}(15) = 8(15) = 120.$$

17. (i) Let $f(x) = (x^2 - 5x + 8)(x^3 + 7x + 9)$ ...(1)

Differentiating (1) w.r.t. x, we get

$$f'(x) = (x^2 - 5x + 8)\frac{d}{dx}(x^3 + 7x + 9) + (x^3 + 7x + 9)\frac{d}{dx}(x^2 - 5x + 8)$$

$\Rightarrow \quad f'(x) = (x^2 - 5x + 8)(3x^2 + 7) + (x^3 + 7x + 9)(2x - 5)$

$\Rightarrow \quad f'(x) = 3x^4 - 15x^3 + 24x^2 + 7x^2 - 35x + 56 + 2x^4$
$+ 14x^2 + 18x - 5x^3 - 35x - 45$

$\Rightarrow \quad f'(x) = 5x^4 - 20x^3 + 45x^2 - 52x + 11.$

(ii) By expanding the product to obtain a single polynomial, we have

$f(x) = (x^2 - 5x + 8)(x^3 + 7x + 9)$

$\Rightarrow \quad f(x) = x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$

$\Rightarrow \quad f(x) = x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$

Differentiating the above equation w.r.t. x, we get

$f'(x) = 5x^4 - 20x^3 + 45x^2 - 52x + 11.$

(iii) By logarithmic differentiation

Let $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

Taking log on both the sides, we get

$\log y = \log\{(x^2 - 5x + 8)(x^3 + 7x + 9)\}$

$\Rightarrow \quad \log y = \log\{(x^2 - 5x + 8) + \log(x^3 + 7x + 9)$

Differentiating the above equation w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{(x^2 - 5x + 8)}(2x-5) + \frac{1}{(x^3+7x+9)}(3x^2+7)$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left\{\frac{2x-5}{x^2-5x+8} + \frac{3x^2+7}{x^3+7x+9}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (x^2-5x+8)(x^3+7x+9)\left[\frac{2x-5}{x^2-5x+8} + \frac{3x^2+7}{x^3+7x+9}\right]$$

$= (2x - 5)(x^3 + 7x + 9) + (3x^2 + 7)(x^2 - 5x + 8)$

$$\frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

$\therefore$ The answer is same in all the three cases.

18. Using product rule : Let $y = u{\cdot}v{\cdot}w = u{\cdot}(vw)$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$\frac{dy}{dx} = u'\cdot(vw) + u\frac{d}{dx}(vw)$$

$= u'\cdot(vw) + u[v'w + vw']$

$= u'vw + uv'w + uvw'$

$$= \frac{du}{dx}\cdot v\cdot w + u\cdot\frac{dv}{dx}\cdot w + u\cdot v\cdot\frac{dw}{dx}$$

Using logarithmic differentiation $= u{\cdot}v{\cdot}w$

Taking log on both sides, we get

$\log y = \log u + \log v + \log w$ ...(ii)

Differentiating (ii) both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$= uvw\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$= vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}$$

$$= \frac{du}{dx}\cdot v\cdot w + u\cdot\frac{dv}{dx}\cdot w + u\cdot v\cdot\frac{dw}{dx}$$

EXERCISE - 5.6

1. Here, $x = 2at^2$ ...(i)

and $y = at^4$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{dt} = 2a(2t) = 4at \text{ and } \frac{dy}{dt} = 4at^3$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2.$$

2. Here, $x = a\cos\theta$ ...(i)

and $y = b\cos\theta$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{d\theta} = a(-\sin\theta) = -a\sin\theta \text{ and}$$

$$\frac{dy}{d\theta} = b(-\sin\theta) = -b\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}.$$

3. Here, $x = \sin t$ ...(i)

and $y = \cos 2t$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\therefore \quad \frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = -\sin 2t\cdot 2 = -2\sin 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{\cos t} = \frac{-2\cdot 2\sin t\cos t}{\cos t}$$

$= -4\sin t$

4. Here, $x = 4t$ ...(i)

$$y = \frac{4}{t} \quad \text{...(ii)}$$

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{dt} = 4 \text{ and } \frac{dy}{dt} = \frac{-4}{t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4}{t^2}\times\frac{1}{4} = \frac{-1}{t^2}.$$

5. Here, $x = \cos\theta - \cos 2\theta$ ...(i)

$y = \sin\theta - \sin 2\theta$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{d\theta} = -\sin\theta - (-\sin 2\theta)\cdot 2 = 2\sin 2\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \cos\theta - \cos 2\theta\cdot 2 = \cos\theta - 2\cos 2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

6. Here, $x = a(\theta - \sin\theta)$ ...(i)

$y = a(1 + \cos\theta)$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{d\theta} = a[1 - \cos\theta]$$

$$\frac{dy}{d\theta} = a[-\sin\theta] = -a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1-\cos\theta)}$$

$$= \frac{-\sin\theta}{1-\cos\theta} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

7. Here, $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$ ...(i)

$y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{dt} = \frac{\sqrt{\cos 2t}\,\frac{d}{dt}\sin^3 t - \sin^3 t\,\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\;3\sin^2 t\cos t - \sin^3 t\cdot\frac{1}{2\sqrt{\cos 2t}}(-\sin 2t)\cdot 2}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\;3\sin^2 t\cos t + \frac{\sin^3 t\sin 2t}{\sqrt{\cos 2t}}}{\cos 2t}$$

$$= \frac{3\cos 2t\sin^2 t\cos t + \sin^3 t\sin 2t}{(\cos 2t)^{3/2}}$$

$$\frac{dy}{dt} = \frac{\sqrt{\cos 2t}\,\frac{d}{dt}\cos^3 t - \cos^3 t\,\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\cdot 3\cos^2 t(-\sin t) - \cos^3 t\cdot\frac{1}{2\sqrt{\cos 2t}}(-\sin 2t)\cdot 2}{\cos 2t}$$

$$= \frac{-3\cos^2 t\cdot\sin t\cdot\sqrt{\cos 2t} + \frac{\cos^3 t\sin 2t}{\sqrt{\cos 2t}}}{\cos 2t}$$

$$= \frac{\cos^3 t\sin 2t - 3\cos^2 t\sin t\cos 2t}{(\cos 2t)^{3/2}}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos^3 t\sin 2t - 3\cos^2 t\cdot\sin t\cos 2t}{3\cos 2t\sin^2 t\cos t + \sin^3 t\sin 2t}$$

8. Here, $x = a\left(\cos t + \log\tan\dfrac{t}{2}\right)$ ...(i)

$y = a\sin t$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}\frac{d}{dt}\tan\frac{t}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}\sec^2\frac{t}{2}\cdot\frac{1}{2}\right] = \frac{a\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t\sin t}{a\cos^2 t} = \tan t$$

9. Here, $x = a\sec\theta$ ...(i)

$y = b\tan\theta$ ...(ii)

Differentiating (i) & (ii) w.r.t. θ, we get

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta \text{ and } \frac{dy}{d\theta} = b\sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$= \frac{b}{a}\frac{\sec\theta}{\tan\theta} = \frac{b}{a}\text{cosec}\,\theta.$$

10. Here, $x = a(\cos\theta + \theta\sin\theta)$ ...(i)

$y = a(\sin\theta - \theta\cos\theta)$ ...(ii)

Differentiating (i) & (ii) w.r.t. θ, we get

$$\frac{dx}{d\theta} = a[-\sin\theta + \theta\cdot\cos\theta + \sin\theta] = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a[\cos\theta - (\theta(-\sin\theta) + \cos\theta)]$$

$= a[\cos\theta + \theta\sin\theta - \cos\theta] = a\theta\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

11. Given, $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$

$$\frac{dx}{dt} = \frac{1}{2}\cdot\frac{1}{\sqrt{a^{\sin^{-1}t}}}\cdot\frac{d}{dt}(a^{\sin^{-1}t})$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{a^{\sin^{-1}t}}}\cdot a^{\sin^{-1}t}\cdot\log a\frac{d}{dt}(\sin^{-1}t)$$

$$= \frac{\sqrt{a^{\sin^{-1}t}}}{2}\cdot\log a\cdot\frac{1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dt} = \frac{1}{2}\cdot\frac{1}{\sqrt{a^{\cos^{-1}t}}}\cdot\frac{d}{dt}(a^{\cos^{-1}t})$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{a^{\cos^{-1}t}}}\cdot a^{\cos^{-1}t}\cdot\log a\cdot\frac{-1}{\sqrt{1-t^2}}$$

$$= \frac{\sqrt{a^{\cos^{-1}t}}}{2}\cdot\log a\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{\sqrt{a^{\cos^{-1}t}}}{2}\cdot\log a\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)}{\frac{\sqrt{a^{\sin^{-1}t}}}{2}\cdot\log a\cdot\frac{1}{\sqrt{1-t^2}}}=\frac{-\sqrt{a^{\cos^{-1}t}}}{\sqrt{a^{\sin^{-1}t}}}=\frac{-y}{x}.$$

EXERCISE - 5.7

1. Let $y = x^2 + 3x + 2$

$$\Rightarrow \frac{dy}{dx}=2x+3 \Rightarrow \frac{d^2y}{dx^2}=2.$$

2. Let $y = x^{20}$

$$\frac{dy}{dx}=20x^{19}$$

$$\frac{d^2y}{dx^2}=20\cdot 19x^{18}=380\,x^{18}.$$

3. Let $y = x\cos x$

$$\frac{dy}{dx}=x\cdot(-\sin x)+\cos x=-x\sin x+\cos x$$

$$\frac{d^2y}{dx^2}=-[x\cos x+\sin x]-\sin x$$

$$= -(x\cos x + 2\sin x)$$

4. Let $y = \log x$

$$\frac{dy}{dx}=\frac{1}{x} \Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{x^2}$$

5. Let $y = x^3 \log x$

$$\frac{dy}{dx}=x^3\cdot\frac{1}{x}+\log x\cdot 3x^2=x^2+3x^2\log x$$

$$\frac{d^2y}{dx^2}=2x+3\left[x^2\cdot\frac{1}{x}+\log x\cdot 2x\right]$$

$= 2x + 3[x + 2x \log x] = 2x + 3x + 6x \log x$

$= 5x + 6x \log x = x\,(5 + 6 \log x)$.

6. Let $y = e^x \sin 5x$

$$\frac{dy}{dx}=e^x\frac{d}{dx}\sin 5x+\sin 5x\frac{d}{dx}e^x$$

$= e^x \cdot \cos 5x \cdot 5 + \sin 5x \cdot e^x = e^x\,[5 \cos 5x + \sin 5x]$

$$\frac{d^2y}{dx^2}=e^x\,[5(-\sin 5x)\cdot 5+\cos 5x\cdot 5]+[5\cos 5x+\sin 5x]\,e^x$$

$= e^x\,[-25 \sin 5x + 5 \cos 5x + 5 \cos 5x + \sin 5x]$

$= e^x\,[10 \cos 5x - 24 \sin 5x] = 2e^x\,[5 \cos 5x - 12 \sin 5x]$

7. Let $y = e^{6x} \cos 3x$

$$\frac{dy}{dx}=e^{6x}(-\sin 3x)\cdot 3+\cos 3x\cdot e^{6x}\cdot 6$$

$= 6 \cdot e^{6x} \cos 3x - 3e^{6x} \sin 3x$

$$\frac{d^2y}{dx^2}=6[e^{6x}(-\sin 3x)\cdot 3+\cos 3x\cdot e^{6x}\cdot 6]$$

$- 3[e^{6x} \cos 3x \cdot 3 + \sin 3x\, e^{6x} \cdot 6]$

$= -18 \cdot e^{6x} \sin 3x + 36\, e^{6x} \cos 3x - 9e^{6x} \cos 3x - 18e^{6x} \sin 3x$

$= 27e^{6x} \cos 3x - 36e^{6x} \sin 3x = 9e^{6x}(3 \cos 3x - 4 \sin 3x)$

8. Let $y = \tan^{-1}x$

$$\frac{dy}{dx}=\frac{1}{(1+x^2)}$$

$$\frac{d^2y}{dx^2}=\frac{(1+x^2)\cdot 0-1\cdot(2x)}{(1+x^2)^2}=\frac{-2x}{(1+x^2)^2}$$

9. Let $y = \log(\log x)$

$$\frac{dy}{dx}=\frac{1}{\log x}\cdot\frac{1}{x}$$

$$\frac{d^2y}{dx^2}=\frac{1}{\log x}\left(-\frac{1}{x^2}\right)+\frac{1}{x}\frac{d}{dx}\left(\frac{1}{\log x}\right)$$

$$=\frac{-1}{x^2\log x}+\frac{1}{x}\left[\frac{\log x\,.\,0-1\cdot\frac{1}{x}}{(\log x)^2}\right]$$

$$=\frac{-1}{x^2\log x}+\frac{1}{x}\left[\frac{-\frac{1}{x}}{(\log x)^2}\right]$$

$$=\frac{-1}{x^2\log x}-\frac{1}{x^2(\log x)^2}=\frac{-1}{x^2\log x}\left[1+\frac{1}{\log x}\right].$$

$$=\frac{-1(1+\log x)}{(x\log x)^2}$$

10. Let $y = \sin(\log x)$

$$\frac{dy}{dx}=\cos(\log x)\cdot\frac{1}{x}$$

$$\frac{d^2y}{dx^2}=\cos(\log x)\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x}\cdot\{-\sin(\log x)\}\frac{1}{x}$$

$$=\frac{-\cos(\log x)}{x^2}-\frac{\sin(\log x)}{x^2}$$

$$=-\frac{1}{x^2}[\cos(\log x)+\sin(\log x)].$$

11. We have $y = 5 \cos x - 3 \sin x$

$$\frac{dy}{dx}=5(-\sin x)-3(\cos x)=-5\sin x-3\cos x$$

$$\frac{d^2y}{dx^2}=-5\cos x-3(-\sin x)=-5\cos x+3\sin x$$

$$\frac{d^2y}{dx^2}+y=-5\cos x+3\sin x+5\cos x-3\sin x=0.$$

12. We have $y = \cos^{-1} x \Rightarrow x = \cos y$ ...(i)

Differentiating (i) w.r.t. x, we get

$$1=-\sin y\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=-\operatorname{cosec} y \quad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\operatorname{cosec} y\cot y\frac{dy}{dx}=-\operatorname{cosec}^2 y\cot y$$

13. We have

$y = 3\cos(\log x) + 4\sin(\log x)$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 3\left[-\sin(\log x)\frac{1}{x}\right] + \left[4\cos(\log x)\frac{1}{x}\right]$$

$$\frac{d^2y}{dx^2} = 3\left[\left(-\sin(\log x)\left(-\frac{1}{x^2}\right)\right) + \frac{1}{x}(-\cos(\log x))\frac{1}{x}\right]$$

$$+4\left[\cos(\log x)\cdot\left(\frac{-1}{x^2}\right) + \frac{1}{x}\{-\sin(\log x)\}\frac{1}{x}\right]$$

$$= \frac{1}{x^2}[3\sin(\log x) - 3\cos(\log x)$$

$$- 4\cos(\log x) - 4\sin(\log x)]$$

$$x^2\frac{d^2y}{dx^2} = [3\sin(\log x) - 4\cos(\log x)$$

$$- (3\cos(\log x) + 4\sin(\log x)]$$

$$x^2\frac{d^2y}{dx^2} = -x\frac{dy}{dx} - y \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

$\Rightarrow$ $x^2y_2 + xy_1 + y = 0$

14. Let $y = Ae^{mx} + Be^{nx}$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = Ae^{mx}\cdot m + Be^{nx}\cdot n = Ame^{mx} + Bne^{nx} \quad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = Ame^{mx}\cdot m + Bne^{nx}\cdot n = Am^2e^{mx} + Bn^2e^{nx} \quad \text{...(iii)}$$

Now, $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = Am^2e^{mx} + Bn^2e^{nx}$

$- [(m + n)(Ame^{mx} + Bne^{nx})] + mn(Ae^{mx} + Be^{nx})$

[From (i), (ii) and (iii)]

$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx}$

$- Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx} = 0$

15. Let $y = 500\,e^{7x} + 600\,e^{-7x}$

$$\frac{dy}{dx} = 500\cdot e^{7x}\cdot 7 + 600\cdot e^{-7x}\cdot(-7)$$

$= 3500\,e^{7x} - 4200\,e^{-7x}$

$$\frac{d^2y}{dx^2} = 3500\cdot 7\cdot e^{7x} - 4200\cdot(-7)e^{-7x}$$

$= 24500\,e^{7x} + 29400\,e^{-7x}$

$= 49\,(500\,e^{7x} + 600\,e^{-7x}) = 49\,y$

$$\therefore \quad \frac{d^2y}{dx^2} = 49\,y$$

16. We have $xe^y + e^y = 1$ ... (i)

Differentiating (i) w.r.t. x, we get

$$xe^y\frac{dy}{dx} + e^y + e^y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-1}{x+1} \quad \text{...(ii)}$$

From (ii), $\left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{x+1}\right)^2$ ...(iii)

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} \quad \text{...(iv)}$$

From (iii) and (iv), we get

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

17. Let $y = (\tan^{-1} x)^2$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1}x\cdot\frac{1}{(1+x^2)}$$

$$\frac{d^2y}{dx^2} = 2\left[\tan^{-1}x\,\frac{(\{1+x^2\}\cdot 0 - 2x)}{(1+x^2)^2} + \frac{1}{(1+x^2)}\cdot\frac{1}{(1+x^2)}\right]$$

$$= 2\left[\frac{-2x\tan^{-1}x}{(1+x^2)^2} + \frac{1}{(1+x^2)^2}\right] = 2\left[\frac{-2x\tan^{-1}x + 1}{(1+x^2)^2}\right]$$

Now, $(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx}$

$$= (x^2+1)^2\cdot 2\left[\frac{-2x\tan^{-1}x + 1}{(1+x^2)^2}\right]$$

$$+ 2x(x^2+1)\cdot 2\tan^{-1}x\,\frac{1}{(1+x^2)}$$

$= -4x\tan^{-1}x + 2 + 4x\tan^{-1}x = 2$

NCERT MISCELLANEOUS EXERCISE

1. Let $y = (3x^2 - 9x + 5)^9$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\Rightarrow \frac{dy}{dx} = 9(3x^2 - 9x + 5)^8\cdot(6x - 9)$$

$= 27(3x^2 - 9x + 5)^8\cdot(2x - 3)$

2. Let $y = \sin^3 x + \cos^6 x$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 3\sin^2 x\cos x + 6\cos^5 x(-\sin x)$$

$= 3\sin x\cos x\,(\sin x - 2\cos^4 x)$

3. Let $y = (5x)^{3\cos 2x}$

Taking log on both sides, we get

$\log y = 3\cos 2x\log(5x) = 3\cos 2x\,[\log 5 + \log x]$

$\log y = 3\cos 2x\log 5 + 3\cos 2x\log x$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = 3\log 5(-\sin 2x)\cdot 2 + \frac{3\cos 2x}{x} + 3\log x(-2\sin 2x)$$

$$= -6\sin 2x\log 5 + \frac{3\cos 2x}{x} - 6\sin 2x\log x$$

$$\frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6(\log 5 + \log x)\sin 2x\right]$$

$$= (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

4. Let $y = \sin^{-1}\left(x\sqrt{x}\right)$ ...(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^3}}\cdot\frac{d}{dx}x\sqrt{x} = \frac{1}{\sqrt{1-x^3}}\left[x\cdot\frac{1}{2\sqrt{x}} + \sqrt{x}\right]$$

$$= \frac{1}{\sqrt{1-x^3}}\left[\frac{\sqrt{x}}{2} + \sqrt{x}\right] = \frac{1}{\sqrt{1-x^3}}\left[\frac{\sqrt{x}+2\sqrt{x}}{2}\right]$$

$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}.$$

5. Let $y = \cos^{-1}\frac{x}{2}(2x+7)^{-1/2}$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \cos^{-1}\frac{x}{2}\left[\frac{d}{dx}(2x+7)^{-1/2}\right] + (2x+7)^{-1/2}\left(\frac{d}{dx}\cos^{-1}\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \cos^{-1}\frac{x}{2}\left[\frac{-1}{2}(2x+7)^{-3/2}(2)\right]$$

$$+ (2x+7)^{-1/2}\left(\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\right)\times\frac{1}{2}$$

$$\frac{dy}{dx} = -\cos^{-1}\frac{x}{2}(2x+7)^{-3/2} + (2x+7)^{-1/2}\left(\frac{-1}{2\sqrt{1-\left(\frac{x^2}{4}\right)}}\right)$$

$$= -\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}} + \frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{3/2}}\right]$$

6. Let $y = \cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}, 0 < x < \frac{\pi}{2}$

$$y = \cot^{-1}\left[\frac{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}} + \sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}} - \sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$y = \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

$$y = \cot^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right]$$

$$y = \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right] \Rightarrow y = \cot^{-1}\left[\cot\frac{x}{2}\right]$$

$$\Rightarrow y = \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

7. Let $y = (\log x)^{\log x}$

Taking log on both sides, we get

$\log y = \log x \log(\log x)$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \log x\cdot\frac{1}{\log x}\cdot\frac{1}{x} + \log(\log x)\cdot\frac{1}{x} = \frac{1}{x}[1+\log(\log x)]$$

$$\frac{dy}{dx} = (\log x)^{\log x}\frac{1}{x}[1+\log(\log x)], x > 1$$

8. Let $y = \cos(a\cos x + b\sin x)$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$\frac{dy}{dx} = -\sin(a\cos x + b\sin x)[a(-\sin x) + b\cos x]$$

$= -\sin(a\cos x + b\sin x)[-a\sin x + b\cos x]$

$= (a\sin x - b\cos x)\sin(a\cos x + b\sin x)$

9. Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

Taking log on both sides, we get

$\log y = (\sin x - \cos x)\log(\sin x - \cos x)$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x)\frac{(\cos x + \sin x)}{(\sin x - \cos x)}$$

$$+ \log(\sin x - \cos x)(\cos x + \sin x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\cos x + \sin x) + \log(\sin x - \cos x)$$

$$\times(\cos x + \sin x)$$

$$\Rightarrow \frac{dy}{dx} = y(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

$$\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}$$

$$\times(\cos x + \sin x)[(1 + \log(\sin x - \cos x))], \sin x > \cos x$$

10. Let $y = x^x + x^a + a^x + a^a$

I II III IV

I. Let $u = x^x \Rightarrow \log u = \log x^x$

$\Rightarrow \log u = x \log x$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\left(\frac{1}{x}\right) + \log x\,(1)$$

$$\Rightarrow \frac{du}{dx} = u\,(1 + \log x)$$

$$\Rightarrow \frac{du}{dx} = x^x\,(1 + \log x) \quad \text{...(ii)}$$

II. Let $v = x^a \Rightarrow \log v = \log x^a$

$\Rightarrow \log v = a \log x$ ...(iii)

Differentiating (iii) both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = a\frac{1}{x} \Rightarrow \frac{dv}{dx} = v\left(\frac{a}{x}\right)$$

$$\Rightarrow \frac{dv}{dx} = x^a\left(\frac{a}{x}\right) = x^a \cdot a \cdot x^{-1} = ax^{a-1} \quad \text{...(iv)}$$

III. Let $w = a^x \Rightarrow \log w = \log a^x$

$\Rightarrow \log w = x \log a$ ...(v)

Differentiating (v) both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{w}\frac{dw}{dx} = \log a(1) \Rightarrow \frac{dw}{dx} = w \log a = a^x \log a \quad \text{...(vi)}$$

IV. Let $z = a^a \Rightarrow \log z = \log a^a$

$\Rightarrow \log z = a \log a$ ...(vii)

Differentiating (vii) both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{z}\frac{dz}{dx} = a(0) \Rightarrow \frac{dz}{dx} = 0 \quad \text{...(viii)}$$

$$\therefore \frac{dy}{dx} = x^x\,(1 + \log x) + ax^{a-1} + a^x \log a$$

[From (ii), (iv), (vi), (viii)]

11. Let $y = x^{x^2 - 3} + (x-3)^{x^2} = u + v$

where $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

Since, $u = x^{x^2-3}$

Taking log on both sides, we get

$\log u = (x^2 - 3) \log x$ ... (i)

Differentiating (i) w.r.t. x on both sides, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{(x^2-3)}{x} + \log x\,(2x)$$

$$\frac{du}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x \log x\right] \quad \text{...(ii)}$$

Also, $v = (x-3)^{x^2}$

Taking log on both sides, we get

$\log v = x^2 \log(x-3)$ ...(iii)

Differentiating (iii) w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \frac{x^2}{x-3} + \log(x-3)\,(2x)$$

$$\frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x \log(x-3)\right] \quad \text{...(iv)}$$

Now, $y = u + v$

$$\text{So, } \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x \log x\right]$$

$$+ (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x \log(x-3)\right] \quad \text{[From (ii) and (iv)]}$$

12. Here $y = 12(1 - \cos t)$ ...(i)

$x = 10\,(t - \sin t)$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dy}{dt} = 12[-(-\sin t)] = 12 \sin t$$

$$\frac{dx}{dt} = 10\,(1 - \cos t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12 \sin t}{10\,(1-\cos t)} = \frac{6 \sin t}{5\,(1 - \cos t)}$$

$$= \frac{6}{5}\left[\frac{2 \sin t/2 \cos t/2}{2\sin^2 t/2}\right] = \frac{6}{5}\cot t/2$$

13. Here $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}$

Let $u = \sin^{-1} x$ and $v = \sin^{-1}\sqrt{1-x^2}$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}, \text{ and } v = \sin^{-1}\sqrt{1-x^2}$$

Put $x = \cos\theta$

$$\therefore v = \sin^{-1}\sqrt{1-\cos^2\theta} = \sin^{-1}\sqrt{\sin^2\theta}$$

$= \sin^{-1}(\sin\theta) = \theta = \cos^{-1}x$

$$\therefore \frac{dv}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\text{As } \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0.$$

14. We have $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x} \Rightarrow x^2(1+y) = y^2(1+x)$$

(Squaring both sides)

$$\Rightarrow (x^2 - y^2) + xy\,(x-y) = 0 \Rightarrow y = \frac{-x}{x+1} \quad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x+1)(-1)-(-x)}{(x+1)^2} = \frac{-x-1+x}{(x+1)^2} = \frac{-1}{(x+1)^2}.$$

15. $(x-a)^2+(y-b)^2=c^2$ ...(i)

Differentiating (i) with respect to x, we get

$$2(x-a)+2(y-b)\frac{dy}{dx}=0$$

$$\Rightarrow\quad (x-a)+(y-b)\frac{dy}{dx}=0 \quad\text{...(ii)}$$

Differentiating (ii) with respect to x, we get

$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

$$\Rightarrow\quad (y-b)\frac{d^2y}{dx^2}=-\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow\quad \frac{d^2y}{dx^2}=-\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{(y-b)} \quad\text{...(iii)}$$

From (ii), we have

$$\frac{dy}{dx}=-\left(\frac{x-a}{y-b}\right)$$

Simplify further,

$$1+\left(\frac{dy}{dx}\right)^2=1+\left(-\left(\frac{x-a}{y-b}\right)\right)^2$$

$$=1+\left(\frac{x-a}{y-b}\right)^2=\frac{c^2}{(y-b)^2} \quad\text{...(iv)}$$

Also, $\frac{d^2y}{dx^2}=-\frac{c^2}{(y-b)^3}$ [From (iii) & (iv)] ...(v)

From (iv) and (v), we have

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\frac{\frac{c^3}{(y-b)^3}}{\frac{-c^2}{(y-b)^3}}=-c$$

which is independent of a and b.

16. We have $\cos y = x\cos(a+y)$

$$\Rightarrow\quad x=\frac{\cos y}{\cos(a+y)}$$

Differentiating the above equation w.r.t. y, we get

$$\frac{dx}{dy}=\frac{\cos(a+y)(-\sin y)-\cos y(-\sin(a+y))}{\cos^2(a+y)}$$

$$=\frac{\cos y\sin(a+y)-\sin y\cos(a+y)}{\cos^2(a+y)}$$

$$=\frac{\sin(a+y-y)}{\cos^2(a+y)}=\frac{\sin a}{\cos^2(a+y)}$$

$$\therefore\quad \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$$

17. Here, $x=a(\cos t+t\sin t)$

Differentiating the above w.r.t. t, we get

$$\frac{dx}{dt}=a(-\sin t+t\cos t+\sin t)$$

$$\frac{dx}{dt}=at\cos t$$

Now, we have, $y=a(\sin t-t\cos t)$

$$\frac{dy}{dt}=a(\cos t-\{t(-\sin t)+\cos t(1)\})$$

$$\frac{dy}{dt}=a(\cos t+t\sin t-\cos t)=at\sin t$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=at\sin t\times\frac{1}{at\cos t}=\tan t$$

$$\frac{d^2y}{dx^2}=\sec^2 t\cdot\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\sec^2 t\cdot\frac{1}{at\cos t}=\frac{\sec^2 t}{at\cos t}=\frac{\sec^3 t}{at}$$

$$\therefore\quad \frac{d^2y}{dx^2}=\frac{\sec^3 t}{at},\ 0<t<\frac{\pi}{2}$$

18. Case I. When $x\ge 0$.

Here $f(x)=|x|^3=x^3$,

$\therefore\quad f'(x)=3x^2$ and $f''(x)=6x$.

Case II. When $x<0$.

Here $f(x)=(-x)^3=-x^3$.

$\therefore\quad f'(x)=-3x^2$ and $f''(x)=-6x$.

$$\text{Thus}\quad f''(x)=\begin{cases}6x, & \text{if } x\ge 0\ ,\\ -6x, & \text{if } x<0\end{cases}$$

Hence $f''(x)=6|x|$.

19. Let $P(n)$ be the given statement in the problem

$$P(n):\frac{d}{dx}(x^n)=nx^{n-1} \quad\text{...(i)}$$

Put $n=1$ in (i), we get

$$P(1):\frac{d}{dx}(x^1)=(1)x^{1-1}=(1)x^0=(1)(1)=1$$

which is true as $\frac{d}{dx}(x)=1$

Suppose $P(m)$ is true

$$P(m):\frac{d}{dx}(x^m)=mx^{m-1} \quad\text{...(ii)}$$

To establish the truth of $P(m+1)$, we prove

$$P(m+1):\frac{d}{dx}(x^{m+1})=(m+1)x^m$$

Now, $x^{m+1} = x^1 \cdot x^m$

$$\frac{d}{dx}(x^{m+1}) = \frac{d}{dx}(x \cdot x^m)$$

$$= x \cdot \frac{d}{dx}(x^m) + x^m \frac{d}{dx}(x) = x \cdot m x^{m-1} + x^m (1)$$

$= mx^m + x^m = x^m(m+1)$

∴ $P(m+1)$ is true if $P(m)$ is true.

∴ By principle of induction $P(n)$ is true for all $n \in N$.

20. We have $\sin(A+B) = \sin A \cos B + \cos A \sin B$ ....(i)

Consider A and B as function of t and differentiating both sides of (i) w.r.t. t, we have

$$\cos(A+B)\left(\frac{dA}{dt} + \frac{dB}{dt}\right) = \sin A(-\sin B)\frac{dB}{dt} + \cos B\left[\cos A \frac{dA}{dt}\right] + \cos A \cos B \frac{dB}{dt} + \sin B(-\sin A)\frac{dA}{dt}$$

$$\Rightarrow \cos(A+B)\left(\frac{dA}{dt} + \frac{dB}{dt}\right) = (\cos A \cos B - \sin A \sin B)\left(\frac{dA}{dt} + \frac{dB}{dt}\right)$$

$\Rightarrow \cos(A+B) = \cos A \cos B - \sin A \sin B$.

21. Let the function be $f(x) = |x-1| + |x-2|$.

We redefine $f(x)$ as :

This is continuous at all $x \in R$ but not differentiable at $x = 1, 2$.

$$f(x) = \begin{cases} -(x-1)-(x-2); \text{ if } x<1 \\ (x-1)-(x-2); \text{ if } 1 \le x \le 2 \\ (x-1)+(x-2); \text{ if } x>2 \end{cases}$$

$$i.e., \quad f(x) = \begin{cases} -2x+3; \text{ if } x<1 \\ 1 \; ; \text{ if } 1 \le x \le 2 \\ 2x-3; \text{ if } x>2 \end{cases}$$

$f(x)$ is clearly continuous at all x except possibly at 1, 2.

At $x = 1$

$$\lim_{x\to1^-} f(x) = \lim_{h\to0}(-2(1-h)+3) = -2+3 = 1$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(1) = 1.$$

Also $f(1) = 1$.

Thus $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$.

Hence $f(x)$ is continuous at $x = 1$.

At $x = 2$

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} 1 = 1$$

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+}(2x-3) = \lim_{h\to0}(2(2+h)-3)$$

$= 2(2) - 3 = 1$.

Also, $f(2) = 1$.

Thus $\lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x) = f(2)$.

Hence $f(x)$ is continuous at $x = 2$.

Hence 'f' is continuous at all $x \in R$.

$$\text{Now } f'(x) = \begin{cases} -2; & \text{if } x<1 \\ 0; & \text{if } 1<x<2 \\ 2; & \text{if } x>2 \end{cases}$$

Derivability at $x = 1$

$$Lf'(1) = \lim_{h\to0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to0^-}\frac{-2(1-h)+3-1}{-h} = \lim_{h\to0^-}\frac{2h}{-h}$$

$$= \lim_{h\to0^-}(-2) = -2.$$

$$Rf'(1) = \lim_{h\to0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0^+}\frac{1-1}{h} = 0.$$

Thus $Lf'(1) \ne Rf'(1)$

$\Rightarrow$ 'f' is not derivable at $x = 1$.

Derivability at $x = 2$

$$Lf'(2) = \lim_{h\to0^-}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to0^-}\frac{1-1}{-h} = 0.$$

$$Rf'(2) = \lim_{h\to0^+}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to0^+}\frac{2(2+h)-3-1}{h} = \lim_{h\to0^+}\frac{2h}{h} = \lim_{h\to0^+} 2 = 2$$

$\Rightarrow Lf'(2) \ne Rf'(2)$

$\Rightarrow f$ is not derivable at $x = 2$

Hence, $f(x) = |x-1| + |x-2|$ is continuous everywhere and differentiable at all $x \in R$ except at 1 and 2.

22. We have $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

$$\therefore \frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx}(f(x)) & \frac{d}{dx}(g(x)) & \frac{d}{dx}(h(x)) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

23. We have $y = e^{a\cos^{-1}x}$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{a\cos^{-1}x}\frac{d}{dx}(a\cos^{-1}x)$$

$$= e^{a\cos^{-1}x}\left(\frac{-a}{\sqrt{1-x^2}}\right) = \frac{-ay}{\sqrt{1-x^2}} \quad \text{...(ii)}$$

Differentiating (ii) both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -a\left[\frac{\sqrt{1-x^2}\frac{dy}{dx} - y\frac{d}{dx}\sqrt{1-x^2}}{(1-x^2)}\right]$$

$$\frac{d^2y}{dx^2} = -a\left[\frac{\sqrt{1-x^2}\frac{dy}{dx} - \frac{y}{2\sqrt{1-x^2}}\cdot(-2x)}{(1-x^2)}\right]$$

$$(1-x^2)\frac{d^2y}{dx^2} = -a\left[-ay + \frac{xy}{\sqrt{1-x^2}}\right] \quad \text{[From (ii)]}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = -a\left[-ay + x\left(\frac{-1}{a}\cdot\frac{dy}{dx}\right)\right]$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = a^2y + x\frac{dy}{dx}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$$