Continuity and Differentiability

CHAPTER 5

TRY YOURSELF

SOLUTIONS

1. Here, we observe that the function $f(x)$ is defined at $x = 1$ and its value is 5.

Now, $\lim_{x\to1} f(x) = \lim_{x\to1}(2x+3) = 2\times1+3 = 5$

Thus, $\lim_{x\to1} f(x) = 5 = f(1)$

Hence, f is continuous at $x = 1$.

2. Here, $f(x)=\begin{cases}\dfrac{x}{\sin 3x}, & \text{when } x\neq 0\\ 3, & \text{when } x=0\end{cases}$

$\therefore \lim_{x\to0} f(x) = \lim_{x\to0}\dfrac{x}{\sin 3x} = \lim_{x\to0}\dfrac{1}{3}\times\dfrac{3x}{\sin 3x} = \dfrac{1}{3}\left(\lim_{3x\to0}\dfrac{3x}{\sin 3x}\right)$

$= \dfrac{1}{3}\times1 = \dfrac{1}{3}$ $\quad\left[\because \lim_{x\to0}\dfrac{\sin x}{x} = 1\right]$

But $f(0) = 3$

$\therefore \lim_{x\to0} f(x) \neq f(0)$

Hence, $f(x)$ is discontinuous at $x = 0$.

3. Here, $f(x)=\begin{cases}5x-4, & \text{when } 0<x\le1\\ 4x^3-3x, & \text{when } 1<x<2\end{cases}$

L.H.L. $= \lim_{x\to1^-} f(x) = \lim_{x\to1^-}(5x-4) = 5(1) - 4 = 1$

R.H.L. $= \lim_{x\to1^+} f(x) = \lim_{x\to1^+}(4x^3-3x) = 4(1)^3 - 3(1) = 1$

and $f(1) = 5\times1-4 = 1$

$\therefore \lim_{x\to1^-} f(x) = f(1) = \lim_{x\to1^+} f(x)$

So, $f(x)$ is continuous at $x = 1$.

4. We have, $f(x) = |x| \Rightarrow f(x) = \begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\ge0\end{cases}$

Clearly, $f(x)$ is defined at $x = 0$ and $f(0) = 0$

L.H.L. $= \lim_{x\to0^-} f(x) = \lim_{x\to0^-}(-x) = \lim_{h\to0}[-(0-h)] = 0$

[Putting $x = 0 - h$]

R.H.L. $= \lim_{x\to0^+} f(x) = \lim_{x\to0^+}(x) = \lim_{h\to0}(0+h) = 0$

[Putting $x = 0 + h$]

$\therefore$ L.H.L. = R.H.L. $\Rightarrow \lim_{x\to0} f(x) = 0$

$\therefore \lim_{x\to0} f(x) = f(0) = 0$

Hence, $f(x)$ is continuous at $x = 0$.

5. Here, $f(x)=\begin{cases}\dfrac{\sin 5x}{3x}, & \text{if } x\neq0\\ k, & \text{if } x=0\end{cases}$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore \lim_{x\to0} f(x) = f(0) \Rightarrow \lim_{x\to0}\left(\dfrac{\sin 5x}{3x}\right) = k$

$\Rightarrow \lim_{x\to0}\dfrac{5}{5}\left(\dfrac{\sin 5x}{3x}\right) = k \Rightarrow \lim_{5x\to0}\dfrac{5}{3}\left(\dfrac{\sin 5x}{5x}\right) = k$

$\Rightarrow \dfrac{5}{3} = k$

6. Here, $f(x) = \dfrac{1}{x}, x\neq0$

Take any non zero real number c.

Clearly, f is defined at every real number c, $c \neq 0$ and $f(c) = \dfrac{1}{c}$.

Also, $\lim_{x\to c} f(x) = \lim_{x\to c}\dfrac{1}{x} = \dfrac{1}{c}$

Thus, $\lim_{x\to c} f(x) = f(c)$ and hence f is continuous at every point in the domain of f.

7. Here, $f(x) = \begin{cases}|x-3|, & \text{if } x\ge1\\ \dfrac{x^2}{4}-\dfrac{3x}{2}+\dfrac{13}{4}, & \text{if } x<1\end{cases}$

Now, if $f(x)$ is continuous at $x = 1$, then $\lim_{x\to1^-} f(x) = f(1)$

At $x = 1$, $f(x) = |x-3| \Rightarrow f(1) = |1-3| = 2$

Now, $\lim_{x\to1^-} f(x) = \lim_{x\to1^-}\left(\dfrac{x^2}{4}-\dfrac{3x}{2}+\dfrac{13}{4}\right)$

$= \lim_{h\to0}\left(\dfrac{(1-h)^2}{4}-\dfrac{3}{2}(1-h)+\dfrac{13}{4}\right)$

$= \dfrac{1}{4}-\dfrac{3}{2}+\dfrac{13}{4} = 2$

$\therefore f(1) = 2 = \lim_{x\to1} f(x)$

$\therefore$ $f(x)$ is continuous at $x = 1$. Thus $f(x)$ is nowhere discontinuous.

8. We have $f(x)=\begin{cases}2 & \text{, if } x\le 3\\ ax+b & \text{, if } 3<x<5\\ 9 & \text{, if } x\ge 5\end{cases}$

When $x<3$, $f(x)=2$ which being a constant functions, is continuous for all $x<3$.
When $3<x<5$, $f(x)=ax+b$, which being a polynomial, is continuous for all $x\in(3,5)$.
When $x>5$, $f(x)=9$, which being a constant function, is continuous for all $x>5$.
We now consider the continuity of $f(x)$ at $x=3$ and $x=5$.
At $x=3$: L.H.L. = $\lim_{x\to3^-}(2)=2$

R.H.L. = $\lim_{x\to3^+}(ax+b)=\lim_{h\to0}[a(3+h)+b]$
$=3a+b$
and $f(3)=2$
$\therefore$ For continuity at $x=3$, $3a+b=2$. ...(i)

At $x=5$: L.H.L. = $\lim_{x\to5^-}(ax+b)=\lim_{h\to0}[a(5-h)+b]$
$=5a+b$
R.H.L. = $\lim_{x\to5^+}(9)=9$
and $f(5)=9$
$\therefore$ For continuity at $x=5$, $5a+b=9$ ...(ii)
Solving (i) and (ii), we get
$a=\frac{7}{2}, b=\frac{-17}{2}$.

9. We have $f(x)=\begin{cases}12x-13\text{, if } x\le3\\ 2x^2+5\text{, if } x>3\end{cases}$

At $x=3$:

L.H.D. $=\lim_{x\to3^-}\frac{f(x)-f(3)}{x-3}=\lim_{x\to3^-}\frac{(12x-13)-23}{x-3}$

$=\lim_{h\to0}\frac{[12(3-h)-13]-23}{(3-h)-3}=\lim_{h\to0}\frac{(36-12h-13-23)}{-h}$

$=\lim_{h\to0}\left(\frac{12h}{h}\right)=12$

R.H.D. $=\lim_{x\to3^+}\frac{f(x)-f(3)}{x-3}=\lim_{x\to3^+}\frac{(2x^2+5)-23}{x-3}$

$=\lim_{h\to0}\frac{(2(3+h)^2+5)-23}{(3+h)-3}=\lim_{h\to0}\frac{[2(9+h^2+6h)+5-23]}{h}$

$=\lim_{h\to0}\left(\frac{18+2h^2+12h-23+5}{h}\right)=\lim_{h\to0}\left(\frac{2h+12}{1}\right)=12$

$\therefore$ L.H.D. = R.H.D.

Hence, $f(x)$ is differentiable at $x=3$.
$\therefore$ $f'(3)=12$

10. We have given, $f(x)=|x-5|$

$\therefore\ f(x)=\begin{cases}-(x-5), & \text{if } x<5\\ x-5, & \text{if } x\ge5\end{cases}$

For continuity at $x=0$

L.H.L. $=\lim_{x\to5^-}f(x)=\lim_{x\to5^-}[-(x-5)]$

$=\lim_{h\to0}[-(5-h)+5]=\lim_{h\to0}h=0$

and R.H.L. $=\lim_{x\to5^+}(x-5)$

$=\lim_{h\to0}(5+h-5)=\lim_{h\to0}h=0$

Also, $f(5)=5-5=0$
L.H.L. = R.H.L. = $f(5)$
Therefore, $f(x)$ is continuous at $x=5$

Now, $Lf'(5)=\lim_{x\to5^-}\frac{f(x)-f(5)}{x-5}$

$=\lim_{x\to5^-}\frac{-x+5-0}{x-5}=\lim_{x\to5^-}\frac{-(x-5)}{x-5}=-1$

$Rf'(5)=\lim_{x\to5^+}\frac{f(x)-f(5)}{x-5}$

$=\lim_{x\to5^+}\frac{x-5-0}{x-5}=1$

Therefore, $Lf'(5)\ne Rf'(5)$
Hence, $f(x)=|x-5|$ is not differentiable at $x=5$.

11. Using definition of derivative, we have

$\lim_{x\to2}\frac{f(x)-f(2)}{x-2}=f'(2)$

$\Rightarrow\ \lim_{x\to2}\frac{f(x)-f(2)}{x-2}=1$ $[\because f'(2)=1]$ ...(i)

Now, $\lim_{x\to2}\frac{xf(2)-2f(x)}{x-2}$

$=\lim_{x\to2}\frac{xf(2)-2f(2)+2f(2)-2f(x)}{x-2}$

$=\lim_{x\to2}\frac{(x-2)f(2)-2(f(x)-f(2))}{x-2}$

$=\lim_{x\to2}\frac{(x-2)f(2)}{x-2}-2\lim_{x\to2}\frac{f(x)-f(2)}{x-2}$

$=f(2)-2f'(2)=4-2\times1=2$ [Using (i) and $f(2)=4$]

12. At $x=c$: L.H.L. = $\lim_{x\to c^-}f(x)=\lim_{x\to c^-}(x-c)\cos\left(\frac{1}{x-c}\right)$

$=\lim_{h\to0}(c-h-c)\cos\left(\frac{1}{c-h-c}\right)=\lim_{h\to0}(-h)\cos\left(\frac{1}{h}\right)$

$=0\times$ an oscillating number between -1 and $1=0$

R.H.L. = $\lim_{x\to c^+}f(x)=\lim_{x\to c^+}(x-c)\cos\left(\frac{1}{x-c}\right)$

$=\lim_{h\to0}(c+h-c)\cos\left(\frac{1}{c+h-c}\right)$

$= \lim_{h\to 0} h\cos\left(\frac{1}{h}\right) = 0 \times$ [an oscillating number between –1 and 1]

$= 0$

Also, $f(c) = 0$

Thus, L.H.L. = R.H.L. = $f(c)$

$\therefore$ $f(x)$ is continuous at $x = 1$.

Differentiability at $x = c$:

$$\text{L.H.D.} = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c^-}\left[\frac{(x-c)\cos\left(\frac{1}{x-c}\right)-0}{x-c}\right]$$

$$= \lim_{h\to 0}\left[\frac{(c-h-c)\cos\left(\frac{1}{c-h-c}\right)}{c-h-c}\right] = \lim_{h\to 0}\cos\left(\frac{1}{h}\right)$$

= a number which oscillates between –1 and 1.

So, L.H.D. does not exist.

Similarly, R.H.D. does not exist.

Hence, $f(x)$ is not differentiable at $x = c$.

13. (i) Let $y = f(x) = \sin(x^2)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin(x^2)) = \cos(x^2)\frac{d}{dx}(x^2)$$

$= 2x \cdot \cos x^2$

(ii) Let $y = f(x) = \cot(\sin\sqrt{x})$

$$\frac{dy}{dx} = \frac{d}{dx}\{\cot(\sin\sqrt{x})\} = \frac{d\{\cot(\sin\sqrt{x})\}}{d(\sin\sqrt{x})}\times\frac{d(\sin\sqrt{x})}{d(\sqrt{x})}\times\frac{d(\sqrt{x})}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\text{cosec}^2(\sin\sqrt{x})\cdot\cos\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

14. Let $y = \frac{1}{\sqrt{a^2-x^2}}$. Putting $u = a^2 - x^2$, we get

$$y = \frac{1}{\sqrt{u}} = u^{-1/2} \text{ and } u = a^2 - x^2$$

$$\therefore \frac{dy}{du} = -\frac{1}{2}u^{-3/2} \text{ and } \frac{du}{dx} = -2x$$

Now, $\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}u^{-3/2}\times(-2x) = -\frac{1}{2u^{3/2}}\times(-2x)$$

$$= \frac{x}{(a^2-x^2)^{3/2}} \quad [\because u = a^2 - x^2]$$

15. Here, $y = \sin^3 5x \cos\sqrt{x}$

Put $\sin^3 5x = u$ and $\cos\sqrt{x} = v$

$$\therefore \frac{du}{dx} = 3\sin^2 5x\cdot\cos 5x\cdot 5 \text{ and } \frac{dv}{dx} = -\sin\sqrt{x}\,.\frac{1}{2\sqrt{x}}$$

$y = u \cdot v$

$$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= \sin^3 5x\left(\frac{-\sin\sqrt{x}}{2\sqrt{x}}\right) + \cos\sqrt{x}\left(15\sin^2 5x\cos 5x\right)$$

$$= \sin^2 5x\left[15\cos\sqrt{x}\cos 5x - \frac{\sin 5x\sin\sqrt{x}}{2\sqrt{x}}\right]$$

16. Let $y = x^4 + \sin 4x + \cos^4 x + \tan x^4$

$$\therefore \frac{dy}{dx} = 4x^3 + \cos 4x\cdot 4 + 4\cos^3 x(-\sin x) + \sec^2 x^4(4x^3)$$

$$= 4x^3 + 4\cos 4x - 4\sin x\cos^3 x + 4x^3\sec^2 x^4$$

17. (i) Here, $y = \cot^{-1}(\sqrt{x}) = \cot^{-1} u$, where $u = \sqrt{x}$

$$\therefore \frac{dy}{dx} = \frac{-1}{1+u^2} \text{ and } \frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = \frac{-1}{1+u^2}\times\frac{1}{2\sqrt{x}} = \frac{-1}{1+(\sqrt{x})^2}\times\frac{1}{2\sqrt{x}}$$

$$= \frac{-1}{2\sqrt{x}(1+x)}$$

(ii) Here, $y = \cot^{-1}(\text{cosec } x + \cot x)$

$$= \cot^{-1}\left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right) = \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}$$

18. Let $y = \tan^{-1}\left\{\sqrt{\frac{1+\sin x}{1-\sin x}}\right\}$.

$$\text{Then, } y = \tan^{-1}\left\{\sqrt{\frac{1-\cos(\pi/2+x)}{1+\cos(\pi/2+x)}}\right\}$$

$$= \tan^{-1}\left\{\sqrt{\frac{2\sin^2(\pi/4+x/2)}{2\cos^2(\pi/4+x/2)}}\right\}$$

$$\Rightarrow y = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \frac{\pi}{4}+\frac{x}{2}$$

$$\left[\because -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{4}+\frac{x}{2} < \frac{\pi}{2}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}$$

19. $\frac{d}{dx}\left\{\tan^{-1}\left(\frac{a+x}{1-ax}\right)\right\}$

$= \frac{d}{dx}\{\tan^{-1} a + \tan^{-1} x\} = \frac{d}{dx}(\tan^{-1} a) + \frac{d}{dx}(\tan^{-1} x)$

$= 0 + \frac{1}{1+x^2} = \frac{1}{1+x^2}$

20. If $y = \cos^{-1}\left(\frac{x-\frac{1}{x}}{x+\frac{1}{x}}\right) = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

$= \cos^{-1}\left(-\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(-\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$ [Putting $x = \tan\theta$]

$= \cos^{-1}(-\cos 2\theta) = \cos^{-1}[\cos(\pi - 2\theta)]$

or $y = \pi - 2\theta = \pi - 2\tan^{-1} x$

$\therefore \frac{dy}{dx} = -\frac{2}{1+x^2}$

21. Here, $x - y = \pi$

Differentiating both sides, we get

$1 - \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = 1$

22. Differentiating both sides of the given relation with respect to x, we get

$3x^2 + 3y^2\frac{dy}{dx} = 3a\left\{x\frac{dy}{dx} + y\right\}$

$\Rightarrow (3y^2 - 3ax)\frac{dy}{dx} = 3ay - 3x^2$

$\Rightarrow 3(y^2 - ax)\frac{dy}{dx} = 3(ay - x^2)$

$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$

23. We have, $(1 + y^2)\sec x - y\cot x + 1 = x^2$

Differentiating both sides w.r.t. x, we get

$(1 + y^2)\sec x\tan x + \sec x \cdot 2y \cdot \frac{dy}{dx} - y(-\text{cosec}^2 x) - \cot x\frac{dy}{dx} = 2x.$

$\Rightarrow \frac{dy}{dx}(2y\sec x - \cot x)$

$= 2x - y\text{cosec}^2 x - (1+y^2)\sec x\tan x$

$\therefore \frac{dy}{dx} = \frac{2x - y\text{cosec}^2 x - (1+y^2)\sec x\tan x}{2y\sec x - \cot x}$

24. We have given, $\tan^{-1}(x^2 + y^2) = a$

On differentiating both sides w.r.t. x, we get

$\frac{d}{dx}\tan^{-1}(x^2+y^2) = \frac{d}{dx}(a)$

$\Rightarrow \frac{1}{1+(x^2+y^2)^2}\cdot\frac{d}{dx}(x^2+y^2) = 0$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

25. (i) Let $y = e^{\sin(\cos x)}$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(e^{\sin(\cos x)}) = e^{\sin(\cos x)}\frac{d}{dx}[\sin(\cos x)]$

$e^{\sin(\cos x)}\cdot\cos(\cos x)\frac{d}{dx}(\cos x)$

$= -\sin x\cdot\cos(\cos x)e^{\sin(\cos x)}$

(ii) Let $y = \cos^{-1}(e^x)$

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\cos^{-1}(e^x)\right] = \frac{-1}{\sqrt{1-e^{2x}}}\frac{d}{dx}(e^x) = \frac{-e^x}{\sqrt{1-e^{2x}}}$

26. Let $y = \sin[\sin(\log 3x)]$

$\therefore \frac{dy}{dx} = \frac{d}{dx}[\sin(\sin(\log 3x)]$

$= \cos[\sin(\log 3x)]\frac{d}{dx}(\sin\log 3x))$

$= \cos[\sin(\log 3x)]\cos(\log 3x)\cdot\frac{d}{dx}(\log 3x)$

$= \cos[\sin(\log 3x)]\cos(\log 3x)\times\frac{1}{3x}\frac{d}{dx}(3x)$

$= \cos[\sin(\log 3x)]\cos(\log 3x)\cdot\frac{1}{x}$

27. $y = \log\sqrt{\frac{1+\cos^2 x}{1-e^{2x}}}$

$= \frac{1}{2}\log\left(\frac{1+\cos^2 x}{1-e^{2x}}\right) = \frac{1}{2}[\log(1+\cos^2 x) - \log(1-e^{2x})]$

$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{1+\cos^2 x}\cdot\frac{d}{dx}(1+\cos^2 x) - \frac{1}{1-e^{2x}}\cdot\frac{d}{dx}(1-e^{2x})\right]$

$= \frac{1}{2}\left[\frac{1}{1+\cos^2 x}\cdot(-2\cos x\sin x) - \frac{1}{1-e^{2x}}(-2e^{2x})\right]$

$= \frac{-\sin x\cos x}{1+\cos^2 x} + \frac{e^{2x}}{1-e^{2x}}.$

28. Let $y = 5^{3-x^2} + (3-x^2)^5$.

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}(5^{3-x^2}) + \frac{d}{dx}\left\{(3-x^2)^5\right\}$

$\Rightarrow \frac{dy}{dx} = 5^{3-x^2}\log_e 5\times\frac{d}{dx}(3-x^2) + 5(3-x^2)^{5-1}\times\frac{d}{dx}(3-x^2)$

$$\Rightarrow \quad \frac{dy}{dx} = 5^{3-x^2}\log_e 5\times(0-2x)+5(3-x^2)^4\times(0-2x)$$

$$\Rightarrow \quad \frac{dy}{dx} = -2x\left\{5^{3-x^2}\log_e 5+5(3-x^2)^4\right\}.$$

29. Let $y = a^x$

Taking logarithm on both sides, we get

$\log y = x \log a$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1\cdot\log a$$

$$\Rightarrow \quad \frac{dy}{dx} = y\log a$$

Thus, $\frac{d}{dx}(a^x) = a^x \log a$

30. Let $y = (\log x)^{\sin x}$.

Then, $y = e^{\sin x \log(\log x)}$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{\sin x\cdot\log(\log x)}\ \frac{d}{dx}\{\sin x\cdot\log(\log x)\}$$

$$= (\log x)^{\sin x}\left\{\log(\log x)\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(\log(\log x))\right\}$$

$$= (\log x)^{\sin x}\left\{\log(\log x)\cdot\cos x+\sin x\times\frac{1}{\log x}\times\frac{1}{x}\right\}$$

Thus, $\frac{dy}{dx} = (\log x)^{\sin x}\left\{\log(\log x)\cdot\cos x+\frac{\sin x}{x\log x}\right\}$

31. Let $y = \frac{8^x}{x^8}$

$$\Rightarrow \quad \log y = \log\frac{8^x}{x^8} = \log 8^x - \log x^8$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}[x\cdot\log 8 - 8\cdot\log x]$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \log 8 - \frac{8}{x}$$

$$\therefore \quad \frac{dy}{dx} = y\left(\log 8-\frac{8}{x}\right) = \frac{8^x}{x^8}\left(\log 8-\frac{8}{x}\right)$$

32. Let $y = x^{x\cos x} + (x\cos x)^x$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = x^{x\cos x}\cdot\frac{d}{dx}(x\cos x\cdot\log x)+(x\cos x)^x\cdot\frac{d}{dx}(x\cdot\log(x\cos x))$$

$$= x^{x\cos x}\left[x\cos x\cdot\frac{1}{x}+\log x.(-x\sin x+\cos x.1)\right]$$

$$+ (x\cos x)^x\left[x.\frac{1}{x\cos x}(-x\sin x+\cos x)+\log(x\cos x).1\right]$$

$= x^{x\cos x}\,[(1 + \log x)\cos x - x\sin x\log x] + (x\cos x)^x\,[\log(x\cos x) - x\tan x + 1]$.

33. Given, $x = a\cos\theta$, $y = a\sin\theta$

$$\therefore \quad \frac{dx}{d\theta} = a(-\sin\theta),\ \frac{dy}{d\theta} = a\cos\theta$$

Hence, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$

34. Here, $x = \cos\theta + \cos 2\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = -\sin\theta - 2\sin 2\theta = -(\sin\theta + 2\sin 2\theta)$$

and $y = \sin\theta + \sin 2\theta$

$$\Rightarrow \quad \frac{dy}{d\theta} = \cos\theta + 2\cos 2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{\cos\theta+2\cos 2\theta}{\sin\theta+2\sin 2\theta}$$

35. We have,

$x = t + \frac{1}{t}$ ...(i)

and $y = t - \frac{1}{t}$ ...(ii)

Taking derivative of (i) and (ii) w.r.t. 't', we get

$$\frac{dx}{dt} = 1-\frac{1}{t^2} = \frac{t^2-1}{t^2} \text{ and } \frac{dy}{dt} = 1+\frac{1}{t^2} = \frac{t^2+1}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(t^2+1)/t^2}{(t^2-1)/t^2} = \frac{t^2+1}{t^2-1}$$

36. Let $y = \tan x$ and $z = \sin x$

$$\Rightarrow \quad \frac{dy}{dx} = \sec^2 x \text{ and } \frac{dz}{dx} = \cos x$$

$$\therefore \quad \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{\sec^2 x}{\cos x} = \sec^3 x.$$

37. Let $u = \tan^{-1}\left(\frac{1+2x}{1-2x}\right)$ and $v = \sqrt{1+4x^2}$.

Then, $u = \tan^{-1}1 + \tan^{-1}2x$ and $v = \sqrt{1+4x^2}$

$$\Rightarrow \quad \frac{du}{dx} = \frac{2}{1+4x^2} \text{ and } \frac{dv}{dx} = \frac{1}{2\sqrt{1+4x^2}}\times 8x = \frac{4x}{\sqrt{1+4x^2}}$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+4x^2}}{\frac{4x}{\sqrt{1+4x^2}}} = \frac{1}{2x\sqrt{1+4x^2}}$$

38. Let $y = x^x$ and $z = x\log x$

Now, $y = x^x \Rightarrow \log y = x\log x$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{x}+1\cdot\log x$$

$$\Rightarrow \quad \frac{dy}{dx} = x^x(1+\log x)$$

Also, $z = x\log x$

$\therefore \quad \frac{dz}{dx} = x \times \frac{1}{x} + \log x \cdot 1 = 1 + \log x$

Hence, $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{x^x(1+\log x)}{1+\log x} = x^x$

39. Given, $y = x^3 + \tan x$

$\therefore \quad \frac{dy}{dx} = 3x^2 + \sec^2 x$ ...(i)

Again differentiating (i) w.r.t x, we get

$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 + \sec^2 x) = 6x + 2\sec x \cdot \sec x \tan x$

$= 6x + 2\sec^2 x \tan x.$

40. Here, $x = 4z^2 + 5 \Rightarrow \frac{dx}{dz} = 8z$

$y = 6z^2 + 7z + 3 \Rightarrow \frac{dy}{dz} = 12z + 7$

$\therefore \quad \frac{dy}{dx} = \frac{dy/dz}{dx/dz} = \frac{12z+7}{8z}$ ...(i)

Differentiating (i) w.r.t. x, we have

$\frac{d^2y}{dx^2} = \frac{(8z)(12) - (12z+7)\cdot 8}{(8z)^2} \cdot \frac{dz}{dx}$

$= \frac{8[12z - 12z - 7]}{(8z)^2} \times \frac{1}{8z} = \frac{-7}{64z^3}$

41. Here, $y = ae^{mx} + be^{-mx}$

$\therefore \quad \frac{dy}{dx} = ame^{mx} - bm\,e^{-mx}$

$\Rightarrow \quad \frac{dy}{dx} = m(ae^{mx} - be^{-mx})$ ...(i)

Differentiating (i) w.r.t. x, we get

$\frac{d^2y}{dx^2} = m(ame^{mx} + bme^{-mx})$

$= m^2(ae^{mx} + be^{-mx}) = m^2y$

Hence, $\frac{d^2y}{dx^2} - m^2y = 0$