CHAPTER 6

Application of Derivatives

NCERT FOCUS

SOLUTIONS

EXERCISE - 6.1

1. Let $A = \pi r^2$ ...(i)
(where A denote the area of the circle when its radius is r),
Differentiating (i), w.r.t. r, we get

$$\Rightarrow \frac{dA}{dr} = \pi(2r) = 2\pi r$$

(a) $\left(\frac{dA}{dr}\right)_{r = 3\text{ cm}} = 2\pi\,(3\text{ cm}) = 6\pi\text{ cm}$

(b) $\left(\frac{dA}{dr}\right)_{r = 4\text{ cm}} = 2\pi\,(4\text{cm}) = 8\pi\text{ cm}$

2. Let at any instant of time t, the edge of the cube be x, surface area be S and the volume be V, then
$V = x^3$ and $S = 6x^2$ ...(i)
Differentiating (i) w.r.t. t, we get

$$\Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt} \quad ...(ii)$$

and, $\frac{dS}{dt} = 6\,(2x)\frac{dx}{dt} = 12x\frac{dx}{dt}$ ...(iii)

$\frac{dV}{dt} = 8\text{ cm}^3/\text{sec}$ (Given)

$\Rightarrow 3x^2\frac{dx}{dt} = 8\text{ cm}^3/\text{sec}$ (Using (ii))

$\Rightarrow 3\,(12\text{ cm})^2\frac{dx}{dt} = 8\text{ cm}^3/\text{sec}$ ($\because x = 12$ cm given)

$\Rightarrow \frac{dx}{dt} = \frac{8}{432}\text{ cm/sec} = \frac{1}{54}\text{ cm/sec}$

Substituting this value of $\frac{dx}{dt}$ in (iii), we get

$\frac{dS}{dt} = 12\,(12\text{ cm})\left(\frac{1}{54}\text{ cm/sec}\right)$ ($\because x = 12$ cm)

$= \frac{8}{3}\text{ cm}^2/\text{sec.}$

3. Let at any instant of time t, the radius of the circle be r and its area be A, then, $A = \pi r^2$

$$\Rightarrow \frac{dA}{dt} = \pi(2r)\frac{dr}{dt}$$

$\therefore \left(\frac{dA}{dt}\right)_{r=10\text{ cm}} = 2\pi$ (10 cm) (3 cm/sec)

$\left(\because\ r = 10\text{ cm and } \frac{dr}{dt} = 3\text{ cm/sec}\right)$

$= 60\,\pi\text{ cm}^2/\text{sec.}$

$\therefore$ Rate of increase of area of the circle is $60\,\pi\text{ cm}^2/\text{sec.}$

4. Let at any instant of time t, the edge of the cube be x and its volume be V, then
$V = x^3$ ...(i)
Differentiating (i) w.r.t. t, we get

$$\Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt} = 3\,(10\text{ cm})^2\,(3\text{ cm/sec})$$

$\left(\because\ x = 10\text{ cm and } \frac{dx}{dt} = 3\text{ cm/sec}\right)$

$= 900\text{ cm}^3/\text{sec.}$

$\therefore$ Rate of increase of volume of cube is $900\text{ cm}^3/\text{sec.}$

5. Let at any instant of time t, the radius of the circular wave be r and the area enclosed be A, then
$A = \pi r^2$ ...(i)
Differentiating (i) w.r.t. t, we have

$$\Rightarrow \frac{dA}{dt} = \pi\,(2r)\frac{dr}{dt} = 2\pi\,(8\text{ cm})\,(5\text{ cm/sec})$$

$\left(\because\ r = 8\text{ cm and } \frac{dr}{dt} = 5\text{ cm/sec}\right)$

$= 80\pi\text{ cm}^2/\text{sec.}$

$\therefore$ Rate of increase of enclosed area $= 80\pi\text{ cm}^2/\text{sec.}$

6. Let at any instant of time t, the radius of the circle be r and its circumference be C, then
$C = 2\pi r$ ...(i)
Differentiating (i) w.r.t. t, we get

$\frac{dC}{dt} = 2\pi\frac{dr}{dt} = 2\pi\,(0.7)\text{ cm/sec}$ $\left(\because\ \frac{dr}{dt} = 0.7\text{ cm/sec}\right)$

$= (1.4\,\pi)$ cm/sec.

Hence, the rate of increase of circumference is $(1.4\,\pi)$ cm/sec.

7. Let any instant of time t, length of rectangle be x, width be y, the perimeter be P and the area be A, then
(a) $P = 2(x + y)$ ...(i)
We have

$\frac{dx}{dt} = -5\text{ cm/min}$ and $\frac{dy}{dt} = 4\text{ cm/min}$

Differentiating (i) w.r.t. t, we get

$$\Rightarrow \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2\,(-5+4)\text{ cm/min}$$

$= -2$ cm/min.

$\therefore$ Perimeter of the rectangle is decreasing at the rate of 2 cm/min.

(b) $A = xy$ ...(ii)
Differentiating (ii) w.r.t. t, we get

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

= (8 cm) (4 cm/min) + (6 cm) (– 5 cm/min)
= 2 cm^2/min.

$\therefore$ Area of the rectangle is increasing at the rate of 2 cm^2/min.

8. Let at any instant of time t, the radius of the balloon be r and its volume be V, then

$$V = \frac{4}{3}\pi r^3 \quad \text{...(i)}$$

Differentiating (i) w.r.t. t, we get

$$\frac{dV}{dt} = \left(\frac{4}{3}\pi\right)\left(3r^2 \frac{dr}{dt}\right)$$

$$\Rightarrow \quad 900 \text{ cm}^3/\text{sec} = \left(\frac{4}{3}\pi\right)\left\{3\,(15\text{ cm})^2 \frac{dr}{dt}\right\}$$

$$\left(\because \frac{dV}{dr} = 900 \text{ cm}^3/\text{sec}\right)$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{900}{4\pi \times (15)^2} \text{ cm/sec} = \frac{1}{\pi} \text{ cm/sec}$$

$\therefore$ Rate of increase of the radius of the balloon $= \frac{1}{\pi}$ cm/sec.

9. Let at any instant of time, the radius of the balloon be r and its volume be V, then

$$V = \frac{4}{3}\pi r^3 \quad \text{...(i)}$$

Differentiating (i) w.r.t. r, we get

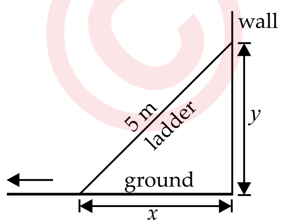
$$\frac{dV}{dr} = \left(\frac{4}{3}\pi\right)3r^2 = 4\pi r^2$$

$$= 4\pi(10 \text{ cm})^2 = 400\,\pi \text{ cm}^2$$

$\therefore$ Rate of increase of volume with respect to change in radius is 400 π cm.

10. If the foot of the ladder is at a distance x from the wall and the top is at a height y at any instant of time t, then,

$$(5\text{ m})^2 = x^2 + y^2 \quad \text{...(i)}$$



Differentiating (i) w.r.t. t, we get

$$\Rightarrow \quad \frac{d}{dt}(25\text{ m}^2) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \text{...(ii)}$$

$$\Rightarrow \quad \frac{dx}{dt} = 2 \text{ cm/s} = 0.02 \text{ m/sec}$$

$$x = 4 \text{ m and } y = \sqrt{25 - 4^2} \text{ m} = 3 \text{ m}$$

$$(\because \; x^2 + y^2 = 25 \Rightarrow y = \sqrt{25 - x^2})$$

Hence, from (ii), we get

$$0 = 2 \times 4 \times 0.02 + 2 \times 3 \frac{dy}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{0.16}{6} \text{ m/sec}$$

$\therefore$ Rate of decrease of height on the wall

$$= \frac{16}{600} \text{ m/sec} = \frac{1600}{600} \text{ cm/sec} = \frac{8}{3} \text{ cm/sec}.$$

11. We have the curve $6y = x^3 + 2$ ...(i)

Differentiating (i) w.r.t. x, we get

$$6\frac{dy}{dx} = 3x^2 \quad \Rightarrow \quad 6 \times 8 = 3x^2$$

$$\left(\because \frac{dy}{dx} = 8 \text{ is given } i.e., \text{ rate of change of } y \text{ co-ordinate w.r.t. } x \text{ co-ordinate is } 8\right)$$

$\Rightarrow \quad x^2 = 16 \quad \Rightarrow \quad x = \pm 4$

When $x = 4 \quad \Rightarrow \quad 6y = 4^3 + 2$

$$\Rightarrow \quad y = \frac{66}{6} = 11$$

When $x = -4 \quad \Rightarrow \quad 6y = (-4)^3 + 2$

$$\Rightarrow \quad y = \frac{-62}{6} = \frac{-31}{3}$$

Hence, the required points are (4, 11) and $\left(-4, \frac{-31}{3}\right)$.

12. Let at any instant of time t, the radius of the bubble be r and its volume be V, then

$$V = \frac{4}{3}\pi r^3 \quad \text{...(i)}$$

Differentiating (i), w.r.t. t, we get

$$\frac{dV}{dt} = \left(\frac{4}{3}\pi\right)\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2 \frac{dr}{dt}$$

$$= 4\pi\,(1\text{ cm})^2\left(\frac{1}{2}\text{ cm/sec}\right) = 2\pi \text{ cm}^3/\text{sec}$$

Hence, the rate of increase of the volume of the bubble = 2π cm^3/sec.

13. Diameter of the balloon, $d = \frac{3}{2}(2x+1)$

$$\therefore \quad \text{Radius of the balloon, } r = \frac{d}{2} = \frac{1}{2}\left\{\frac{3}{2}(2x+1)\right\}$$

$$= \frac{3}{4}(2x+1)$$

So, the volume V of the balloon

$$V = \frac{4}{3}\pi\,(\text{radius})^3 = \frac{4}{3}\pi\left\{\frac{3}{4}(2x+1)\right\}^3 = \frac{9\pi}{16}(2x+1)^3 \quad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$\frac{dV}{dx} = \frac{9\pi}{16} \times 3\,(2x+1)^2 \times 2 = \frac{27\pi}{8}(2x+1)^2$$

14. Let at any instant of time t, the radius of the base of the cone be r, its height be h and the volume of the sand cone be V, then

$$h = \frac{1}{6}r \quad \Rightarrow \quad r = 6h$$

and $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi(6h)^2 h=12\pi h^3$ ...(i)

Differentiating (i) w.r.t. t, we get

$$\frac{dV}{dt}=(12\pi)\left(3h^2\frac{dh}{dt}\right)$$

$$\Rightarrow \quad 12\text{ cm}^3/\text{sec}=36\pi(4\text{ cm})^2\frac{dh}{dt}$$

$$\left(\because\ h=4\text{ cm and }\frac{dV}{dt}=12\text{ cm}^3/\text{sec}\right)$$

$$\Rightarrow \quad \frac{dh}{dt}=\frac{12}{36\pi\times16}\text{ cm/sec}=\frac{1}{48\pi}\text{ cm/sec}$$

$\therefore$ Rate of increase of the height of the sand cone $=\frac{1}{48\pi}$ cm/sec.

15. We have, $C(x)=0.007x^3-0.003x^2+15x+4000$ ...(i)

Differentiating (i) w.r.t. x, we get

Marginal cost $=\frac{dC}{dx}$

$=\frac{d}{dx}\{0.007x^3-0.003x^2+15x+4000\}$

$=0.007\times3x^2-0.003\times2x+15+0$

$\left(\frac{dC}{dx}\right)_{x=17}=0.007\times3(17)^2-0.003\times2\times17+15$

$=20.967$

$\therefore$ Marginal cost (when $x=17$) is 20.967.

16. We have $R(x)=13x^2+26x+15$ ...(i)

Differentiating (i) w.r.t. x, we get

Marginal revenue,

$\frac{dR}{dx}=\frac{d}{dx}(13x^2+26x+15)$

$=13\times2x+26=26x+26$

$\therefore \quad \left(\frac{dR}{dx}\right)_{x=7}=26\times7+26=208$

$\Rightarrow$ Marginal revenue (when $x=7$) is ₹ 208.

17. (B) : If A is the area of the circle corresponding to radius r, then

$A=\pi r^2$ ...(i)

Differentiating (i) w.r.t. r, we get

$\frac{dA}{dr}=2\pi r\quad\left(\frac{dA}{dr}\right)_{r=6\text{ cm}}=2\pi(6)=12\pi$

18. (D) : We have $R(x)=3x^2+36x+5$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dR}{dx}=\frac{d}{dx}(3x^2+36x+5)=6x+36$$

$$\Rightarrow \quad \left(\frac{dR}{dx}\right)_{x=15}=6\times15+36=126$$

EXERCISE - 6.2

1. We have $f(x)=3x+17$ ...(i)

$f(x)$ being a polynomial function, is continuous and derivable on R.

Differentiating (i), w.r.t. x, we get

$f'(x)=3>0\ \forall\ x\in R,$

$\Rightarrow$ f is increasing on R.

2. We have $f(x)=e^{2x}$ ...(i)

$f(x)$ is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$f'(x)=e^{2x}\cdot2=2e^{2x}>0$ for all $x\in R$

$\Rightarrow$ f is increasing on R.

3. We have $f(x)=\sin x$ ...(i)

which is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$f'(x)=\cos x$

(a) For all $x\in\left(0,\frac{\pi}{2}\right)$, $\cos x>0$, therefore,

$f(x)=\sin x$ is strictly increasing in $\left(0,\frac{\pi}{2}\right)$

(b) Since, $\cos x<0$ for all $x\in\left(\frac{\pi}{2},\pi\right)$, therefore $f(x)=\sin x$ is strictly decreasing in $\left(\frac{\pi}{2},\pi\right)$

(c) From parts (a) and (b), we conclude that $f(x)=\sin x$ is neither increasing nor decreasing on $(0,\pi)$.

4. We have $f(x)=2x^2-3x$ ...(i)

$f(x)$ is a polynomial function. Hence $f(x)$ is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$f'(x)=4x-3.$

(a) For increasing

$$f'(x)>0\ \Rightarrow\ 4x-3>0\ \Rightarrow\ x>\frac{3}{4}$$

$\therefore$ f is increasing on $\left(\frac{3}{4},\infty\right)$.

(b) For decreasing

$$f'(x)<0\ \Rightarrow\ 4x-3<0\ \Rightarrow\ x<\frac{3}{4}$$

$\therefore$ f is strictly decreasing on $\left(-\infty,\frac{3}{4}\right)$.

5. We have $f(x)=2x^3-3x^2-36x+7$ ...(i)

$f(x)$ is a polynomial function. Hence $f(x)$ is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$$f'(x)=\frac{d}{dx}(2x^3-3x^2-36x+7)$$

$=6x^2-6x-36$

$=6(x^2-x-6)=6(x-3)(x+2)$

(a) For increasing, $f'(x) > 0$

$\Rightarrow$ $6(x-3)(x+2) > 0$

$\Rightarrow$ $(x-3)(x-(-2)) > 0 \Rightarrow x < -2$ or $x > 3$

$\Rightarrow$ $x \in (-\infty, -2) \cup (3, \infty)$.

$\therefore$ f is increasing on $(-\infty, -2) \cup (3, \infty)$.

(b) For decreasing, $f'(x) < 0$

$\Rightarrow$ $6(x-3)(x+2) < 0$

$\Rightarrow$ $(x-3)(x-(-2)) < 0 \Rightarrow x < 3$ or $x > -2$

$\Rightarrow$ $-2 < x < 3 \Rightarrow x \in (-2, 3)$

$\therefore$ f is decreasing on $(-2, 3)$.

6. (a) We have $f(x) = x^2 + 2x - 5$ ...(i)

$f(x)$ being a polynomial, is continuous and derivable on R.

Differentiating (i), w.r.t. x, we get

$f'(x) = 2x + 2$

For increasing, $f'(x) > 0$

$\Rightarrow 2x + 2 > 0 \Rightarrow x > -1$

For decreasing, $f'(x) < 0$

$\Rightarrow$ $2x + 2 < 0 \Rightarrow x < -1$

$\therefore$ $f(x)$ is strictly decreasing for $x < -1$

$f(x)$ is strictly increasing for $x > -1$

(b) We have $f(x) = 10 - 6x - 2x^2$ ...(i)

$f(x)$ being a polynomial, is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$f'(x) = 0 - 6 - 2(2x) = -6 - 4x$

For increasing, $f'(x) > 0$

$\Rightarrow$ $-6 - 4x > 0 \Rightarrow -4x > 6$

$\Rightarrow$ $x < -\dfrac{3}{2} \Rightarrow x \in \left(-\infty, -\dfrac{3}{2}\right)$

$\therefore$ $f(x)$ is strictly increasing if $x < -\dfrac{3}{2}$

For decreasing, $f'(x) < 0$

$\Rightarrow$ $-6 - 4x < 0 \Rightarrow -4x < 6$

$\Rightarrow$ $-x < \dfrac{3}{2} \Rightarrow x > -\dfrac{3}{2} \Rightarrow x \in \left(-\dfrac{3}{2}, \infty\right)$

$\therefore$ $f(x)$ is strictly decreasing if $x > -\dfrac{3}{2}$

(c) We have $f(x) = -2x^3 - 9x^2 - 12x + 1$ ...(i)

$f(x)$ being a polynomial, is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$f'(x) = -2 \times 3x^2 - 9(2x) - 12$

$= -6x^2 - 18x - 12 = -6(x^2 + 3x + 2)$

$= -6(x+1)(x+2)$

For increasing/decreasing, put $f'(x) = 0$

$\Rightarrow$ $-6(x+1)(x+2) = 0 \Rightarrow x = -1, x = -2$

For increasing and decreasing, we have the following intervals

Interval	Sign of $f'(x)$ $= -6(x+1)(x+2)$	Nature of function
$-\infty < x < -2$	(–ve) (–ve) (–ve) = –ve	f is strictly decreasing
$-2 < x < -1$	(–ve) (–ve) (+ve) = +ve	f is strictly increasing
$-1 < x < \infty$	(–ve) (+ve) (+ve) = –ve	f is strictly decreasing

(d) We have $f(x) = 6 - 9x - x^2$ ...(i)

$f(x)$ being a polynomial, is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$f'(x) = -9 - 2x$.

For increasing, $f'(x) > 0$

$\Rightarrow -9 - 2x > 0 \Rightarrow -2x > 9$

$\Rightarrow$ $x < -\dfrac{9}{2} \Rightarrow x \in \left(-\infty, -\dfrac{9}{2}\right)$

$\therefore$ f is strictly increasing on $\left(-\infty, -\dfrac{9}{2}\right)$.

For decreasing, $f'(x) < 0$

$\Rightarrow -9 - 2x < 0 \Rightarrow -2x < 9$

$\Rightarrow$ $x > -\dfrac{9}{2} \Rightarrow x \in \left(-\dfrac{9}{2}, \infty\right)$

$\therefore$ f is strictly decreasing on $\left(-\dfrac{9}{2}, \infty\right)$.

(e) We have $f(x) = (x+1)^3 (x-3)^3$ ...(i)

$f(x)$ being polynomial, is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$f'(x) = (x+1)^3 \cdot 3(x-3)^2 \cdot 1 + (x-3)^3 \cdot 3(x+1)^2 \cdot 1$

$= (x-3)^2 (x+1)^2 \{3(x+1) + 3(x-3)\}$

$= (x-3)^2 (x+1)^2 \{6x - 6\}$

$= 6(x-3)^2 (x+1)^2 (x-1)$

For increasing/decreasing, put $f'(x) = 0$

$\Rightarrow$ $6(x-3)^2(x+1)^2(x-1) = 0$

$\Rightarrow$ $x = 3, x = -1, x = 1$

For increasing and decreasing, we have the following intervals.

Interval	Sign of $f'(x)$ $= 6(x-3)^2 (x+1)^2$ $(x-1)$	Nature of function
$-\infty < x < -1$	(+ve) (+ve) (+ve) (–ve) = –ve	f is strictly decreasing
$-1 < x < 1$	(+ve) (+ve) (+ve) (–ve) = –ve	f is strictly decreasing
$1 < x < 3$	(+ve) (+ve) (+ve) (+ve) = +ve	f is strictly increasing
$3 < x < \infty$	(+ve) (+ve) (+ve) (+ve) = +ve	f is strictly increasing

7. We have $y = \log(1+x) - \dfrac{2x}{2+x}, x > -1$ ...(i)

Here, $\dfrac{dy}{dx} = \dfrac{1}{1+x} - 2\left\{\dfrac{(2+x)\cdot 1 - x(0+1)}{(2+x)^2}\right\}$

$= \dfrac{1}{1+x} - \dfrac{4}{(2+x)^2} = \dfrac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$

$= \dfrac{x^2}{(1+x)(2+x)^2} \ \forall\ x > -1$

$x^2 > 0$, $(2 + x)^2 > 0$ (being perfect squares)
and $(1 + x) > 0\ \ \forall\, x > -1$

$\Rightarrow\ \dfrac{dy}{dx} \ge 0$ for all $x > -1$

Hence y is an increasing function of x throughout its domain.

8. We have $y = [(x(x - 2)]^2, x \in R$

$y = (x^2 - 2x)^2$ ...(i)

Differentiating (i) w.r.t. x, we get

$\Rightarrow\ \dfrac{dy}{dx} = 2(x^2 - 2x)\dfrac{d}{dx}(x^2 - 2x)$

$= 2(x^2 - 2x)(2x - 2) = 4x(x - 1)(x - 2)$

For increasing/decreasing, put $\dfrac{dy}{dx} = 0$

$\Rightarrow\ 4x(x - 1)(x - 2) = 0\ \Rightarrow\ x = 0, 1, 2.$

We divide the points 0, 1, 2 in the following intervals.

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
sign of x	–ve	+ve	+ve	+ve
sign of $(x-1)$	–ve	–ve	+ve	+ve
sign of $(x-2)$	–ve	–ve	–ve	+ve
sign of dy/dx	–ve	+ve	–ve	+ve
Nature of function	decreasing	increasing	decreasing	increasing

$\therefore$ y is an increasing function in $(0, 1) \cup (2, \infty)$.

9. We have $y = \dfrac{4\sin\theta}{2+\cos\theta} - \theta,\ \theta \in \left[0, \dfrac{\pi}{2}\right]$.

$\dfrac{dy}{d\theta} = 4\left\{\dfrac{(2+\cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(2+\cos\theta)^2}\right\} - 1$

$= \dfrac{4(2\cos\theta + 1)}{(2+\cos\theta)^2} - 1\ = \dfrac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2}$

$= \dfrac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2} = \dfrac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$

$\cos\theta > 0$ in $\left[0, \dfrac{\pi}{2}\right]$; $4 - \cos\theta > 0$ in $\left[0, \dfrac{\pi}{2}\right]$

$\left(\because\ -1 \le \cos\theta \le 1,\ \text{if}\ \ \theta \in \left[0, \dfrac{\pi}{2}\right]\right)$,

$(2+\cos\theta)^2 > 0$ in $\left[0, \dfrac{\pi}{2}\right]$

(being a perfect square)

$\Rightarrow\ \dfrac{dy}{d\theta} > 0$ for all $\theta \in \left[0, \dfrac{\pi}{2}\right]$

$\Rightarrow$ y is strictly increasing function in $\left[0, \dfrac{\pi}{2}\right]$.

Also, y is continuous at $x = 0$ and $x = \pi/2$.
Hence, y is increasing in $[0, \pi/2]$.

10. We have $f(x) = \log x$ ...(i)

Domain of f is $(0, \infty)$

Now, $f'(x) = \dfrac{1}{x} > 0$ for all $x \in (0, \infty)$

$\Rightarrow\ f'(x) > 0$ for all $x \in (0, \infty)$

$\therefore$ f is increasing on $(0, \infty)$

(Note that $\log x$ is defined only for $x > 0$)

11. We have $f(x) = x^2 - x + 1, x \in (-1, 1)$ ...(i)

Differentiating (i) w.r.t. x, we get

$f'(x) = 2x - 1$

For increasing, $f'(x) > 0$

$\Rightarrow\ 2x - 1 > 0\ \Rightarrow\ x > \dfrac{1}{2}$

For decreasing, $f'(x) < 0$

$\Rightarrow\ 2x - 1 < 0\ \Rightarrow\ x < \dfrac{1}{2}$

$\Rightarrow\ f'(x) > 0$ for all $x \in \left(\dfrac{1}{2}, 1\right)$

and $f'(x) < 0$ for all $x \in \left(-1, \dfrac{1}{2}\right)$

Hence, f is neither increasing nor decreasing on $(-1, 1)$.

12. (A,B) : (A) Let $f(x) = \cos x$, then

$f'(x) = -\sin x < 0$ for all $x \in \left(0, \dfrac{\pi}{2}\right)$

$\left[\because\ \sin x > 0\ \forall\ x \in \left(0, \dfrac{\pi}{2}\right)\right]$

$\Rightarrow$ f is decreasing on $(0, \pi/2)$.

(B) Let $f(x) = \cos 2x$, then

$f'(x) = -2\sin 2x < 0$ for all $x \in \left(0, \dfrac{\pi}{2}\right)$ $(\because\ \sin 2x > 0)$

$\Rightarrow$ f is decreasing on $\left(0, \dfrac{\pi}{2}\right)$.

(C) Let $f(x) = \cos 3x$, then $f'(x) = -3\sin 3x$,
which assumes +ve as well as –ve values in $\left(0, \dfrac{\pi}{2}\right)$.

$\therefore$ f is neither increasing nor decreasing on $\left(0, \dfrac{\pi}{2}\right)$.

(D) Let $f(x) = \tan x$, then $f'(x) = \sec^2 x > 0$ for all $x \in \left(0, \dfrac{\pi}{2}\right)$

$\left[\because \sec^2 x > 0\ \forall\ x \in \left(0, \dfrac{\pi}{2}\right)\right]$

$\Rightarrow$ f is increasing on $\left(0, \dfrac{\pi}{2}\right)$.

Thus, we find that the function in (a) and (b) are decreasing on $\left(0, \dfrac{\pi}{2}\right)$.

13. (D) : We have $f(x) = x^{100} + \sin x - 1$ ...(i)
Differentiating (i) w.r.t. x, we get

$f'(x) = 100x^{99} + \cos x$

(A) $f'(x)$ assumes only +ve values in (0, 1)
$\therefore$ f is increasing (0, 1)

(B) $f'(x) > 0$ for all $x \in \left(\frac{\pi}{2}, \pi\right)$, therefore f is increasing in $x \in \left(\frac{\pi}{2}, \pi\right)$

(C) $f'(x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$, therefore f is increasing in $x \in \left(0, \frac{\pi}{2}\right)$

In all the above cases, we find that $f'(x)$ is not decreasing in any of the intervals. Hence, (d) is the correct option.

14. We have, $f(x) = x^2 + ax + 1$ ...(i)
$\Rightarrow$ $f'(x) = 2x + a$
If $1 < x < 2 \Rightarrow 2 < 2x < 4$
$\Rightarrow$ $2 + a < 2x + a < 4 + a$
$\Rightarrow$ $2 + a < f'(x) < 4 + a$
Now, $f(x)$ is increasing on (1, 2) only if $f'(x) > 0$ for $2x + a > 0 \Rightarrow x > -a/2$
$\therefore$ We have to find the least value of a such that $x > \frac{-a}{2}$, when $x \in (1, 2)$
Thus, the least value of a for f to be increasing on (1, 2) is

$$-\frac{a}{2} = 1 \Rightarrow a = -2$$

$\therefore$ Required least value of a is –2.

15. We have $f(x) = x + \frac{1}{x}, x \in I$ ...(i)
Differentiating (i) w.r.t. x, we get

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

$x^2 > 0$ in (–1, 1), $x^2 - 1 > 0 \Rightarrow x^2 > 1$
$\Rightarrow$ $x < -1$ or $x > 1$
$\Rightarrow$ $x \in (-\infty, -1)$ or $x \in (1, \infty)$
$\Rightarrow$ $x \in (-\infty, -1) \cup (1, \infty)$
$\Rightarrow$ $x \in R - [-1, 1]$
$\Rightarrow$ $f(x)$ is increasing on I
($\because$ I is an interval which is a subset of $R - [-1, 1]$)

16. We have $f(x) = \log(\sin x)$ ...(i)
Differentiating (i) w.r.t. x, we get

$$f'(x) = \frac{1}{\sin x}(\cos x) = \cot x$$

As $\cot x > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$ and $\cot x < 0$ for all $x \in \left(\frac{\pi}{2}, \pi\right)$, therefore, $f(x)$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

17. We have $f(x) = \log(\cos x)$ ...(i)
Differentiating (i) w.r.t. x, we get

$$f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$$

As $\tan x > 0 \Rightarrow -\tan x < 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$ and $\tan x < 0$
$\Rightarrow -\tan x > 0$ for all $x \in \left(\frac{\pi}{2}, \pi\right)$, therefore, $f'(x) < 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$ and $f'(x) > 0$ for all $x \in \left(\frac{\pi}{2}, \pi\right)$.
Hence $f(x)$ is decreasing on $\left(0, \frac{\pi}{2}\right)$ and increasing on $\left(\frac{\pi}{2}, \pi\right)$.

18. We have $f(x) = x^3 - 3x^2 + 3x - 100$ ...(i)
Differentiating (i) w.r.t. x, we get
$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1)$
$= 3(x - 1)^2 \geq 0$ for all $x \in R$
$\Rightarrow$ $3 > 0, (x - 1)^2 \geq 0$ (being perfect square)
$\Rightarrow$ $f'(x) \geq 0 \Rightarrow f(x)$ is increasing on R.

19. (D) : We have, $y = x^2 e^{-x}$ ...(i)
Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = x^2 e^{-x}(-1) + e^{-x}(2x)$$

$= xe^{-x}(-x + 2) = x(2 - x)e^{-x}$

For increasing/decreasing, put $\frac{dy}{dx} = 0$
$\Rightarrow$ $x(2 - x)e^{-x} = 0 \Rightarrow x = 0, x = 2$
The above points divided into the following intervals

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
sign of x	–ve	+ve	+ve
sign of $(2 - x)$	+ve	+ve	–ve
sign of e^{-x}	+ve	+ve	+ve
sign of $f'(x)$	–ve	+ve	–ve

Hence, $f(x)$ is increasing in (0, 2).

EXERCISE - 6.5

1. (i) We have, $f(x) = (2x - 1)^2 + 3$, for all $x \in R$.
Since, $(2x - 1)^2 \geq 0 \Rightarrow (2x - 1)^2 + 3 \geq 3$
$\therefore$ Minimum $f(x) = 3$, which occurs when $2x - 1 = 0$ *i.e.*, when $x = 1/2$.
Value of $f(x)$ has no maximum value, because $f(x) \to \infty$ as $|x| \to \infty$.
(ii) We have,

$$f(x) = 9x^2 + 12x + 2 = 9\left(x^2 + \frac{4}{3}x\right) + 2$$

$$= 9\left\{x^2 + \frac{4}{3}x + \frac{4}{9}\right\} + 2 - 4 = 9\left(x + \frac{2}{3}\right)^2 - 2$$

Since, $\left(x+\frac{2}{3}\right)^2 \geq 0 \Rightarrow 9\left(x+\frac{2}{3}\right)^2 - 2 \geq -2$

$\Rightarrow$ $f(x) \geq -2$ for all $x \in R$.

$\therefore$ Minimum $f(x) = -2$, which occurs when $x+\frac{2}{3}=0$, *i.e.*, when $x=\frac{-2}{3}$.

and $f(x)$ has no maximum value, because, $f(x) \to \infty$ as $|x| \to \infty$.

(iii) We have, $f(x) = 10 - (x-1)^2$ for all $x \in R$.

Since $(x-1)^2 \geq 0 \ \forall \ x \in R$

$\Rightarrow$ $-(x-1)^2 \leq 0 \ \forall \ x \in R$

$\Rightarrow$ $10 - (x-1)^2 \leq 10 \ \forall \ x \in R$

$\therefore$ Maximum $f(x) = 10$, which occurs when $x - 1 = 0$ *i.e.*, when $x = 1$.

and $f(x)$ has no minimum value, because, $f(x) \to -\infty$ as $|x| \to \infty$.

(iv) We have $g(x) = x^3 + 1$.

As $x \to \infty$, $g(x) \to \infty$ and as $x \to -\infty$, $g(x) \to -\infty$

$\therefore$ $g(x)$ has neither a maximum nor a minimum value.

2. (i) We have, $f(x) = |x+2| - 1$ for all $x \in R$.

$|x+2| \geq 0, \ |x+2| - 1 \geq -1$

$\therefore$ Minimum $f(x) = -1$, which occurs when $x + 2 = 0$ *i.e.*, when $x = -2$.

and $f(x)$ has no maximum value, because, $f(x) \to \infty$ as $|x| \to \infty$.

(ii) We have $g(x) = -|x+1| + 3$ for all $x \in R$.

$|x+1| \geq 0 \Rightarrow -|x+1| \leq 0$

$\Rightarrow$ $-|x+1| + 3 \leq 3$

$\therefore$ Maximum value of $g(x) = 3$, which occurs when $x + 1 = 0$, *i.e.*, when $x = -1$.

Note that $g(x)$ has no minimum value, because, $g(x) \to -\infty$ as $|x| \to \infty$.

(iii) We have $h(x) = \sin 2x + 5 \ \forall \ x \in R$

We know that $-1 \leq \sin 2x \leq 1$ for all $x \in R$

$\Rightarrow$ $5 - 1 \leq 5 + \sin 2x \leq 5 + 1$ for all $x \in R$

$\Rightarrow$ $4 \leq f(x) \leq 6$ for all $x \in R$.

$\therefore$ Maximum value of $f(x) = 6$, which occurs when $\sin 2x = 1$, and minimum value of $f(x) = 4$, which occurs when $\sin 2x = -1$.

(iv) We have, $f(x) = |\sin 4x + 3| \ \forall \ x \in R$.

We know that $-1 \leq \sin 4x \leq 1$ for all $x \in R$.

$\Rightarrow$ $3 - 1 \leq \sin 4x + 3 \leq 1 + 3 \ \forall \ x \in R$.

$\Rightarrow$ $|2| \leq |\sin 4x + 3| \leq |4| \ \forall \ x \in R$ ($\because \sin 4x + 3 \geq 0 \ \forall \ x \in R$)

$\Rightarrow$ $|2| \leq f(x) \leq |4| \ \forall \ x \in R$

$\therefore$ Minimum value of $f(x) = 2$, which occurs when $\sin 4x = -1$, and maximum value of $f(x) = 4$, which occurs when $\sin 4x = 1$.

(v) We have $h(x) = x + 1, \ -1 < x < 1$.

$-1 < x < 1 \Leftrightarrow -1 + 1 < x + 1 < 1 + 1 \Leftrightarrow 0 < x + 1 < 2$

Here, range of f is $(0, 2)$.

$\therefore$ f has neither a maximum nor a minimum value.

3. (i) We have, $f(x) = x^2 \Rightarrow f'(x) = 2x$

For maximum/minimum or critical points,

$f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$.

Now, $f''(x) = 2 \Rightarrow f''(0) = 2 > 0$

$\therefore$ f has a local minima at $x = 0$ and local minimum value is $f(0) = 0^2 = 0$.

(ii) We have, $g(x) = x^3 - 3x \Rightarrow g'(x) = 3x^2 - 3$

For critical points, $g'(x) = 0$

$3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1, -1$.

The points at which extremum may occurs are -1 and 1.

$g''(x) = 6x$

$g''(-1) = 6(-1) = -6 < 0$

$\therefore$ g has a local maximum at $x = -1$ and local maximum value at $x = -1$ is

$g(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$

$g''(1) = 6 \times 1 = 6 > 0$

$\therefore$ g has a local minimum at $x = 1$ and local minimum value at $x = 1$ is

$g(1) = 1^3 - 3 \times 1 = 1 - 3 = -2$

(iii) We have $h(x) = \sin x + \cos x$, $0 < x < \frac{\pi}{2}$

$\Rightarrow$ $h'(x) = \cos x - \sin x$, for all $x \in \left(0, \frac{\pi}{2}\right)$

For critical points, $h'(x) = 0$

$\Rightarrow$ $\cos x - \sin x = 0 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$

The point at which extremum may occurs is $x = \frac{\pi}{4}$

$\Rightarrow$ $h''(x) = -\sin x - \cos x$, for all $x \in \left(0, \frac{\pi}{2}\right)$

$$h''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} < 0.$$

$\therefore$ h has a local maximum at $x = \frac{\pi}{4}$ and local maximum value at $x = \frac{\pi}{4}$ is

$$h\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

(iv) We have $f(x) = \sin x - \cos x$, $0 < x < 2\pi$

$\Rightarrow$ $f'(x) = \cos x + \sin x$

For critical points, $f'(x) = 0$

$\Rightarrow$ $\cos x + \sin x = 0 \Rightarrow \tan x = -1$

$\Rightarrow$ $x = \pi - \frac{\pi}{4}, \ 2\pi - \frac{\pi}{4} \Rightarrow x = \frac{3\pi}{4}, \ \frac{7\pi}{4}$

$\therefore$ The points at which extremum may occur are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

$f''(x) = -\sin x + \cos x$

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} < 0$$

$\therefore$ f has local maximum at $x = \frac{3\pi}{4}$ and local maximum value at $x = \frac{3\pi}{4}$ is

$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

and $f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$

$= -\left(-\sin\frac{\pi}{4}\right) + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$

$\therefore$ f has local minimum at $x = \frac{7\pi}{4}$ and local minimum value at $x = \frac{7\pi}{4}$ is

$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

(v) We have, $f(x) = x^3 - 6x^2 + 9x + 15, x \in R$

$\Rightarrow$ $f'(x) = 3x^2 - 12x + 9, x \in R$

For critical points, $f'(x) = 0$

$\Rightarrow$ $3x^2 - 12x + 9 = 0 \Rightarrow 3(x-1)(x-3) = 0$

$\Rightarrow$ $x = 1, x = 3.$

$\therefore$ The points where extremum may occurs are 1 and 3.

$f''(x) = 6x - 12, x \in R.$

$f''(1) = 6 - 12 = -6 < 0$

$\therefore$ f has a local maximum at $x = 1$ and local maximum value at $x = 1$ is

$f(1) = 1 - 6 + 9 + 15 = 19$

and $f''(3) = 6 \times 3 - 12 = 6 > 0$

$\therefore$ f has a local minimum at $x = 3$ and local minimum value at $x = 3$ is

$f(3) = 3^3 - 6 \times 3^2 + 9 \times 3 + 15 = 15$

(vi) Given, $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

$\Rightarrow$ $g'(x) = \frac{1}{2} + \left(-\frac{2}{x^2}\right), x > 0$

For critical points, $g'(x) = 0$

$\Rightarrow$ $\frac{1}{2} - \frac{2}{x^2} = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = -2, 2$

But $x > 0$

$\therefore$ The only point where extremum may occur is 2.

$g''(x) = -2(-2)x^{-3}, x > 0$ and $g''(2) = 4(2)^{-3} = \frac{4}{8} = \frac{1}{2} > 0$

g has a local minimum at $x = 2$ and local minimum value is $g(2) = \frac{2}{2} + \frac{2}{2} = 2$.

(vii) Given, $g(x) = \frac{1}{x^2+2}$,

$\Rightarrow$ $g'(x) = \frac{-2x}{(x^2+2)^2}$

For critical points, $g'(x) = 0$

$\Rightarrow$ $\frac{-2x}{(x^2+2)^2} = 0 \Rightarrow x = 0$ $\quad (\because x^2 + 2 \neq 0)$

$g''(x) = \frac{6x^2 - 4}{(x^2+2)^3}$; $g''(0) = \frac{-4}{8} < 0$

$\therefore$ g has a local maximum at $x = 0$ and local maximum value is $g(0) = \frac{1}{0+2} = \frac{1}{2}$.

(viii) We have, $f(x) = x\sqrt{1-x}, x > 0$

$\Rightarrow$ $f'(x) = \frac{x(-1)}{2\sqrt{1-x}} + \sqrt{1-x} = \frac{-x + 2(1-x)}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$

For critical points, $f'(x) = 0$

$\Rightarrow$ $\frac{2-3x}{2\sqrt{1-x}} = 0 \Rightarrow x = \frac{2}{3}.$

$\therefore$ The point at which extremum may occur is 2/3.

$$f''(x) = \frac{\frac{1}{2}\left\{(\sqrt{1-x})(-3) - \frac{(2-3x)(-1)}{2\sqrt{1-x}}\right\}}{(1-x)}$$

$$= \frac{1}{2}\left[\frac{2(1-x)(-3) + 2 - 3x}{2\sqrt{1-x}(1-x)}\right]$$

$$f''\left(\frac{2}{3}\right) = \frac{1}{2}\left[\frac{2\left(1-\frac{2}{3}\right)(-3) + 2 - 3\times\left(\frac{2}{3}\right)}{2\sqrt{1-\frac{2}{3}}\left(1-\frac{2}{3}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{-2\times1\times3}{3} + 2 - 2}{2\sqrt{\frac{1}{3}}\left(\frac{1}{3}\right)}\right] < 0$$

$\therefore$ f has local maximum at $x = \frac{2}{3}$ and local maximum value is

$f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{\frac{1}{3}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$

4. (i) We have, $f(x) = e^x \Rightarrow f'(x) = e^x \ \forall\, x \in R$

$f'(x) = e^x > 0 \ \forall\, x \in R$

$\Rightarrow$ f has no critical point.

Thus, there is no point at which f may have an extremum.

$\therefore$ f has neither maxima nor minima.

(ii) We have $g(x) = \log x, x > 0$

$\Rightarrow$ $g'(x) = \frac{1}{x}, x > 0$

$g'(x) = \frac{1}{x} \neq 0$ for all $x \in (0, \infty)$

$\Rightarrow$ g has no critical point.

Thus, there is no point at which g may have an extremum.

$\therefore$ g has neither maxima nor minima.

(iii) We have $h(x) = x^3 + x^2 + x + 1, x \in R.$

$\Rightarrow$ $h'(x) = 3x^2 + 2x + 1, x \in R$

For critical points, $h'(x) = 0$

$\Rightarrow$ $3x^2 + 2x + 1 = 0 \Rightarrow x = \frac{-2 \pm \sqrt{-8}}{6}$

which is non-real.

$\Rightarrow$ h has no critical point.

Thus, there is no point at which h may have an extremum.

$\therefore$ h has neither maxima nor minima.

5. (i) $f(x) = x^3, x \in [-2, 2] \Rightarrow f'(x) = 3x^2$

For critical points, $f'(x) = 0$

$\Rightarrow$ $3x^2 = 0 \Rightarrow x = 0 \in [-2, 2]$

Hence, for finding the absolute maximum value and the absolute minimum value, we have to evaluate $f(0)$, $f(-2)$ and $f(2)$.

Now, $f(0) = 0^3$, $f(-2) = (-2)^3 = -8$ and $f(2) = 2^3 = 8$

$\therefore$ Absolute maximum value of $f(x) = 8$ at $x = 2$ and absolute minimum value of $f(x) = -8$ at $x = -2$.

(ii) We have $f(x) = \sin x + \cos x,\ x \in [0, \pi]$

$\Rightarrow\ f'(x) = \cos x - \sin x$

For critical points, let $f'(x) = 0$

$\Rightarrow\ \cos x - \sin x = 0 \Rightarrow \tan x = 1$

$\Rightarrow\ x = \frac{\pi}{4} \in [0, \pi]$

Hence, for finding the absolute maximum and the absolute minimum, we have to evaluate $f(0)$, $f(\pi)$ and $f\left(\frac{\pi}{4}\right)$.

Now, $f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$,

$f(\pi) = \sin \pi + \cos \pi = 0 - 1 = -1$

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$\therefore$ Absolute maximum value of $f(x) = \sqrt{2}$ at $x = \frac{\pi}{4}$

Absolute minimum value of $f(x) = -1$ at $x = \pi$.

(iii) We have, $f(x) = 4x - \frac{1}{2}x^2,\ x \in \left[-2, \frac{9}{2}\right]$

$\Rightarrow\ f'(x) = 4 - x$

For critical points, let $f'(x) = 0$

$\Rightarrow\ 4 - x = 0 \Rightarrow x = 4 \in [-2, 9/2]$

Hence, for finding the absolute maximum and the absolute minimum of $f(x)$, we have to evaluate $f(-2)$, $f(9/2)$ and $f(4)$.

Now, $f(-2) = 4(-2) - \frac{1}{2}(-2)^2 = -8 - 2 = -10$,

$$f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2 = 18 - \frac{81}{8} = \frac{63}{8} \text{ and}$$

$$f(4) = 4 \times 4 - \frac{1}{2}(4)^2 = 16 - 8 = 8$$

$\therefore$ Absolute maximum value of $f(x) = 8$ at $x = 4$

Absolute minimum value of $f(x) = -10$ at $x = -2$.

(iv) We have, $f(x) = (x - 1)^2 + 3,\ x \in [-3, 1]$

$\Rightarrow\ f'(x) = 2(x - 1)$

For critical points, $f'(x) = 0$

$\Rightarrow\ 2(x - 1) = 0 \Rightarrow x = 1 \in [-3, 1]$

Hence, for finding the absolute maximum and the absolute minimum of $f(x)$, we have to evaluate $f(-3)$ and $f(1)$.

Now, $f(-3) = (-3 - 1)^2 + 3 = 19$ and $f(1) = (1 - 1)^2 + 3 = 3$

$\therefore$ Absolute maximum value of $f(x) = 19$ at $x = -3$

Absolute minimum value of $f(x) = 3$ at $x = 1$.

6. We have, $p(x) = 41 - 72x - 18x^2$

$p'(x) = -72 - 36x$

Now, for critical points, $p'(x) = 0$

$-72 - 36x = 0 \Rightarrow x = -2$

$p''(x) = -36 < 0$

$\therefore$ Profit is maximum at $x = -2$, and maximum profit is

$p(-2) = 41 - 72(-2) - 18(-2)^2 = 41 + 144 - 72$

$= 185 - 72 = 113$

7. We have, $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25,\ x \in [0, 3]$

$\Rightarrow\ f'(x) = 12x^3 - 24x^2 + 24x - 48$

$= 12[x^3 - 2x^2 + 2x - 4]$

$= 12[x^2(x - 2) + 2(x - 2)] = 12(x - 2)(x^2 + 2)$

For critical points, $f'(x) = 0$

$\Rightarrow\ 12(x - 2)(x^2 + 2) = 0 \Rightarrow x - 2 = 0$

$\Rightarrow\ x = 2 \qquad (\because\ x^2 + 2 \geq 2\ \forall\ x \in R)$

So, to find the maximum and minimum value we have to evaluate $f(0)$, $f(2)$ and $f(3)$

Now, $f(0) = 25$

$f(2) = 3 \times 2^4 - 8 \times 2^3 + 12 \times 2^2 - 48 \times 2 + 25 = -39$

$f(3) = 3 \times 3^4 - 8 \times 3^3 + 12 \times 3^2 - 48 \times 3 + 25 = 16$

$\therefore$ Maximum value of $f(x) = 25$ at $x = 0$ and minimum value of $f(x) = -39$ at $x = 2$.

8. We have, $f(x) = \sin 2x,\ 0 \leq x \leq 2\pi$

$\Rightarrow\ f'(x) = 2\cos 2x$

Let $f'(x) = 0 \Rightarrow 2\cos 2x = 0 \Rightarrow \cos 2x = 0$

$\Rightarrow\ 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \qquad (\because\ 0 < x < 2\pi \Leftrightarrow 0 < 2x < 4\pi)$

$\Rightarrow\ x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

So, for finding the maximum and minimum value, we have to evaluate

$f(0), f(2\pi), f\left(\frac{\pi}{4}\right), f\left(\frac{3\pi}{4}\right), f\left(\frac{5\pi}{4}\right)$ and $f\left(\frac{7\pi}{4}\right)$.

Now, $f(0) = \sin(2 \times 0) = 0$, $f(2\pi) = \sin(2 \times 2\pi) = 0$,

$$f\left(\frac{\pi}{4}\right) = \sin\left(2 \times \frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1,$$

$$f\left(\frac{3\pi}{4}\right) = \sin\left(2 \times \frac{3\pi}{4}\right) = \sin\frac{3\pi}{2} = \sin\left(\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1,$$

$$f\left(\frac{5\pi}{4}\right) = \sin\left(2 \times \frac{5\pi}{4}\right) = \sin\frac{5\pi}{2} = \sin\frac{\pi}{2} = 1$$

and $f\left(\frac{7\pi}{4}\right) = \sin\left(2 \times \frac{7\pi}{4}\right) = \sin\frac{7\pi}{2} = -\sin\frac{\pi}{2} = -1$

$\therefore$ Maximum $f(x) = 1$ at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

9. Let $f(x) = \sin x + \cos x,\ x \in R$

$f'(x) = \cos x - \sin x$

For critical points, $f'(x) = 0$

$\Rightarrow\ \cos x - \sin x = 0 \Rightarrow \tan x = 1$

$\Rightarrow\ x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \sqrt{2}$$

$$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right) + \cos\left(\pi + \frac{\pi}{4}\right)$$

$$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\sqrt{2}$$

$\therefore$ Maximum value of the function is $\sqrt{2}$ at $x = \frac{\pi}{4}$.

10. Let $f(x) = 2x^3 - 24x + 107,\ 1 \leq x \leq 3$

$\Rightarrow\ f'(x) = 6x^2 - 24$

For critical points, $f'(x) = 0$

$\Rightarrow\ 6x^2 - 24 = 0 \Rightarrow 6(x^2 - 4) = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

If we consider the function in the interval [1, 3], then we take $x = 2 \in [1, 3]$

So, for maximum and minimum value, we have to evaluate $f(1), f(2)$ and $f(3)$

Now, $f(1) = 2 \times 1^3 - 24 \times 1 + 107 = 85$

$f(2) = 2 \times 2^3 - 24 \times 2 + 107 = 75$

$f(3) = 2 \times 3^3 - 24 \times 3 + 107 = 89$

$\therefore$ Maximum $f(x) = 89$ at $x = 3$

If we consider the function in the interval [–3, –1], then we take $x = -2 \in [-3, -1]$

So, we evaluate $f(-1), f(-2)$, and $f(-3)$.

$f(-1) = 2(-1)^3 - 24(-1) + 107 = 129,$

$f(-2) = 2(-2)^3 - 24(-2) + 107 = 139,$

$f(-3) = 2(-3)^3 - 24(-3) + 107 = 125$

$\therefore$ Maximum $f(x) = 139$ at $x = -2$.

11. Let $f(x) = x^4 - 62x^2 + ax + 9, 0 \le x \le 2$

$\Rightarrow\ f'(x) = 4x^3 - 124x + a$

We have, $f(x)$ attains maximum value at $x = 1 \in [0, 2]$.

$\therefore$ We must have $f'(1) = 0 \Rightarrow 4 - 124 + a = 0$

$\Rightarrow\ a = 120$

12. Let $f(x) = x + \sin 2x, 0 \le x \le 2\pi$

$\Rightarrow\ f'(x) = 1 + 2\cos 2x$

For critical points, $f'(x) = 0$

$$\Rightarrow\ 1 + 2\cos 2x = 0 \Rightarrow \cos 2x = -\frac{1}{2}$$

$$\Rightarrow\ \cos 2x = -\cos\frac{\pi}{3}$$

(If $0 < x < 2\pi$, then $0 < 2x < 4\pi$)

$$\Rightarrow\ \cos 2x = \cos\left(\pi - \frac{\pi}{3}\right), \cos\left(\pi + \frac{\pi}{3}\right), \cos\left(3\pi - \frac{\pi}{3}\right), \cos\left(3\pi + \frac{\pi}{3}\right)$$

$$2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$$

$$\Rightarrow\ x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

So, for finding maximum and minimum, we evaluate $f(x)$ at $0, 2\pi, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

Now, $f(0) = 0 + \sin 0 = 0,$

$f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi,$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \sin\left(\pi - \frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2},$$

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} + \sin\left(\pi + \frac{\pi}{3}\right) = \frac{2\pi}{3} - \sin\frac{\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2},$$

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \sin\left(2\pi + \frac{2\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{2\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

and $$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin\frac{10\pi}{3} = \frac{5\pi}{3} + \sin\left(3\pi + \frac{\pi}{3}\right) = \frac{5\pi}{3} - \sin\frac{\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$$

Thus, maximum value of $f(x) = 2\pi$ at $x = 2\pi$ and minimum value of $f(x) = 0$ at $x = 0$.

13. Let the one positive number be x

$\therefore$ Another number $= 24 - x$.

Let $p = x(24 - x) \Rightarrow p = 24x - x^2$

$$\frac{dp}{dx} = 24 - 2x$$

For p to be largest,

$$\frac{dp}{dx} = 0 \Rightarrow 24 - 2x = 0 \Rightarrow x = 12$$

and $$\frac{d^2p}{dx^2} = -2, \left(\frac{d^2p}{dx^2}\right)_{x=12} = -2 < 0$$

$\Rightarrow\ p$ has a maximum value at $x = 12$.

So, the required numbers are 12 and 24 – 12 *i.e.* 12

14. We have, two numbers x, and y and

$x + y = 60$ ...(i)

Let $P = xy^3 \Rightarrow P = (60 - y)y^3$ (From (i))

$$\Rightarrow\ P = 60y^3 - y^4 \Rightarrow \frac{dP}{dy} = 180y^2 - 4y^3$$

For maximum P, we must have $\frac{dP}{dy} = 0$

$\Rightarrow\ 180y^2 - 4y^3 = 0 \Rightarrow 4y^2(45 - y) = 0$

$\Rightarrow\ y = 45$ $(\because 0 < y < 60)$

Also, $\frac{d^2P}{dy^2} = 360y - 12y^2$ and

$$\left(\frac{d^2P}{dy^2}\right)_{y=45} = 360 \times 45 - 12 \times (45)^2 < 0$$

Therefore, P is maximum when $y = 45$.

$\therefore$ Required numbers are $y = 45$.

and $x = 60 - y = 60 - 45 = 15$

15. We have, two numbers x and y and let $P = x^2y^5$ and

$x + y = 35$ ...(i)

$\Rightarrow\ P = (35 - y)^2y^5$ (From (i))

$$\text{Now, } \frac{dP}{dy} = (35 - y)^2(5y^4) + y^5 \times 2(35 - y)(-1)$$

$= y^4(35 - y)\,\{5(35 - y) - 2y\} = y^4(35 - y)(175 - 7y)$

For maximum P, $\frac{dP}{dy} = 0$

$\Rightarrow\ y^4(35 - y)(175 - 7y) = 0$

$\Rightarrow\ 175 - 7y = 0$ $(\because 0 < y < 35)$

$\Rightarrow\ y = 25$

Now, $\frac{d^2P}{dy^2} = 4(35-y)(175-7y)y^3 + y^4(-1)(175-7y)$
$+ y^4(35-y)(-7)$

$\Rightarrow \left(\frac{d^2P}{dy^2}\right)_{y=25} < 0$

$\therefore$ P has a maximum value at $y = 25$.

$\therefore$ Required numbers are $y = 25$ and $x = 35 - 25 = 10$

16. Let one number be x

$\therefore$ Another number $= 16 - x$

$S = x^3 + (16 - x)^3$

$\Rightarrow \frac{dS}{dx} = 3x^2 + 3(16-x)^2(-1)$

For minimum S, $\frac{dS}{dx} = 0$

$\Rightarrow$ $3x^2 - 3(16 - x)^2 = 0$

$\Rightarrow$ $x^2 - (256 + x^2 - 32x) = 0$

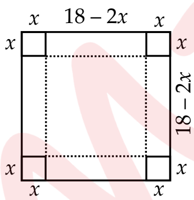
$\Rightarrow$ $32x = 256 \Rightarrow x = 8$

$\frac{d^2S}{dx^2} = 6x + 6(16-x)$ and $\left(\frac{d^2S}{dx^2}\right)_{x=8} = 96 > 0.$

$\therefore$ S has a minimum at $x = 8$

$\therefore$ The required numbers are 8 and 8.

17. Let x cm be the length of each side of the square which is to be cut off from each corner of the square tin sheet of side 18 cm.



Let V be the volume of the open box formed by folding up the flaps, then

$V = x(18 - 2x)(18 - 2x) = 4x(9 - x)^2$

$= 4(x^3 - 18x^2 + 81x)$ ...(i)

Differentiate (i) w.r.t. x, we get

$\frac{dV}{dx} = 4(3x^2 - 36x + 81) = 12(x^2 - 12x + 27)$

For maximum/minimum volume,

$\frac{dV}{dx} = 0 \Rightarrow 12(x^2 - 12x + 27) = 0$

$\Rightarrow$ $12(x - 3)(x - 9) = 0$

$\Rightarrow$ $x = 3, 9$ but $0 < x < 9 \Rightarrow x = 3$

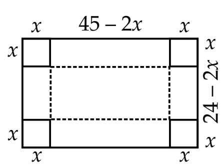
$\frac{d^2V}{dx^2} = 12(2x - 12) = 24(x - 6)$

and $\left(\frac{d^2V}{dx^2}\right)_{x=3} = 24(3-6) = -72 < 0$

$\Rightarrow$ V is maximum at $x = 3$

Hence, the volume of the box is maximum when the side of the square to be cut off is 3 cm.

18. Let the side of the square to be cut from each of the four corners be x cm, then the base of the box has dimensions $(45 - 2x)$ cm and $(24 - 2x)$ cm and height of the box is x cm.



Let V cm^3 be the corresponding volume of the box, then

$V = x(24 - 2x)(45 - 2x)$

$= x(4x^2 - 138x + 1080)$

$= 4x^3 - 138x^2 + 1080x$

$\Rightarrow \frac{dV}{dx} = 12x^2 - 276x + 1080$

For maximum/minimum volume,

$\frac{dV}{dx} = 0 \Rightarrow 12x^2 - 276x + 1080 = 0$

$\Rightarrow$ $x^2 - 23x + 90 = 0 \Rightarrow (x - 18)(x - 5) = 0$

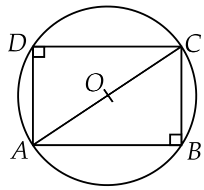
$\Rightarrow$ $x = 5, x \neq 18$ $(\because 0 < x < 12)$

$\frac{d^2V}{dx^2} = 24x - 276$ and $\left(\frac{d^2V}{dx^2}\right)_{x=5} = 24 \times 5 - 276 < 0$

$\therefore$ V is maximum at $x = 5$.

So, a square of side 5 cm should be cut from each corner of the box to have a maximum volume.

19. Let $ABCD$ be a rectangle inscribed in the given circle of radius r having centre at O.



Let one side of the rectangle be x, then the other side

$= \sqrt{(2r)^2 - x^2} = \sqrt{4r^2 - x^2}$

($\because$ $\angle ADC = \angle ABC = 90°$; an angle in the semi-circle).

Let A be the corresponding area of the rectangle, then

$A = x\sqrt{4r^2 - x^2}$, $0 < x < 2r$

$\Rightarrow \frac{dA}{dx} = \frac{x(-2x)}{2\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2} = \frac{2(2r^2 - x^2)}{\sqrt{4r^2 - x^2}}$

For maximum/minimum area

$\frac{dA}{dx} = 0 \Rightarrow 2\left(\frac{2r^2 - x^2}{\sqrt{4r^2 - x^2}}\right) = 0 \Rightarrow x = \sqrt{2}r$ $(\because 0 < x < 2r)$

Now, $\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}(-4x) - (4r^2 - 2x^2)\frac{1 \times (-2x)}{2\sqrt{4r^2 - x^2}}}{(4r^2 - x^2)}$

$= \frac{(4r^2 - x^2)(-4x) + (4r^2 - 2x^2)x}{(4r^2 - x^2)^{3/2}} = \frac{-12r^2x + 2x^3}{(4r^2 - x^2)^{3/2}}$

$\left(\frac{d^2A}{dx^2}\right)_{x=\sqrt{2}r} = \frac{-12r^2(\sqrt{2}r) + 2(\sqrt{2}r)^3}{(2r^2)^{3/2}}$

$= \frac{4\sqrt{2}r^3 - 12\sqrt{2}r^3}{2\sqrt{2}r^3} = -4 < 0$

$\therefore$ Area is maximum at $x = \sqrt{2}r$.

$\therefore$ Length of rectangle $= \sqrt{2}r$

and width of rectangle $= \sqrt{4r^2 - x^2} = \sqrt{2}r$

Hence, the rectangle is a square of side $\sqrt{2}r$ for maximum area.

20. Let r be the radius of the circular base, h be the height and S be the total surface area of a right circular cylinder, then $S = 2\pi r^2 + 2\pi rh$.

Let V be the volume of cylinder with the above dimensions, then

$\therefore$ $V = \pi r^2 h = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right)$

$\left(\because S = 2\pi r^2 + 2\pi rh, \therefore h = \frac{S - 2\pi r^2}{2\pi r}\right)$

$$= \frac{r}{2}(S - 2\pi r^2)$$

$$\Rightarrow \quad V = \frac{Sr}{2} - \pi r^3 \quad \text{...(i)}$$

Differentiating (i), w.r.t. r, we get

$$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$$

For maximum/minimum volume,

$$\frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} - 3\pi r^2 = 0 \Rightarrow r^2 = \frac{S}{6\pi}$$

$$\Rightarrow \quad r = \sqrt{\frac{S}{6\pi}}$$

Now, $\frac{d^2V}{dr^2} = -6\pi r$

and $\left(\frac{d^2V}{dr^2}\right)_{r=\sqrt{S/(6\pi)}} = -6\pi\sqrt{\frac{S}{6\pi}} < 0$

$\Rightarrow$ V has a maximum value at $r = \sqrt{\frac{S}{6\pi}}$.

When $r = \sqrt{\frac{S}{6\pi}}$, then $h = \frac{S - 2\pi\left(\frac{S}{6\pi}\right)}{2\pi\sqrt{\frac{S}{6\pi}}} = \frac{4\pi S/6\pi}{2\pi\sqrt{\frac{S}{6\pi}}}$

$\Rightarrow$ $h = 2\sqrt{\frac{S}{6\pi}} = 2$ radius = diameter.

So, volume is maximum when the height is equal to the diameter of the base.

21. Let r cm be the radius, h cm be the height, S cm^2 be the total surface area and V cm^3 be the volume.

Now, $V = \pi r^2 h = 100 \Rightarrow h = \frac{100}{\pi r^2}$ ...(i)

and $S = 2\pi r^2 + 2\pi rh$ ...(ii)

$$\Rightarrow \quad S = 2\pi r^2 + 2\pi r\left(\frac{100}{\pi r^2}\right) = 2\pi r^2 + \frac{200}{r} \quad \text{(Using (i))}$$

$$\Rightarrow \quad \frac{dS}{dr} = 4\pi r - \frac{200}{r^2}$$

For maximum/minimum surface area,

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r - \frac{200}{r^2} = 0$$

$$\Rightarrow \quad r^3 = \frac{200}{4\pi} \Rightarrow r = \left(\frac{50}{\pi}\right)^{1/3}$$

$$\frac{d^2S}{dr^2} = 4\pi + \frac{200 \times 2}{r^3} = 4\pi + \frac{400}{r^3}$$

and $\left(\frac{d^2S}{dr^2}\right)_{r=(50/\pi)^{1/3}} = 4\pi + \frac{400}{(50/\pi)} > 0$

$\therefore$ S has a minimum value at $r = \left(\frac{50}{\pi}\right)^{1/3}$.

When $r = \left(\frac{50}{\pi}\right)^{1/3}$ cm, then $h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{2/3}} = \frac{100}{(50)^{2/3}\pi^{1/3}}$

$$\Rightarrow \quad h = \frac{50 \times 2}{(50)^{2/3}\pi^{1/3}} = 2\left(\frac{50}{\pi}\right)^{1/3} \text{ cm.}$$

22. Let the length of one piece be x m and other piece is of length $(28 - x)$ m

Let the length of the piece bent into the shape of a circle be x m and length of the other piece bent into the shape of a square is $(28 - x)$ m.

Circumference $= 2\pi r \Rightarrow 2\pi r = x \Rightarrow r = \frac{x}{2\pi}$

$\Rightarrow$ Area of the circle $= \pi(\text{radius})^2 = \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$.

Now, perimeter of square = 4 side

$\Rightarrow$ $28 - x = 4$ side $\Rightarrow$ side $= \frac{28 - x}{4}$

$\Rightarrow$ Area of the square $= (\text{side})^2 = \left(\frac{28 - x}{4}\right)^2 = \frac{(28 - x)^2}{16}$.

Let A be the sum of the areas of the two figures, then

$$A = \frac{x^2}{4\pi} + \frac{(28 - x)^2}{16} \quad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$\frac{dA}{dx} = \frac{2x}{4\pi} + \frac{2(28 - x)(-1)}{16} = \frac{x}{2\pi} - \frac{28 - x}{8}$$

For maximum/minimum, $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{x}{2\pi} - \frac{28 - x}{8} = 0 \Rightarrow \frac{4x - 28\pi + x\pi}{8\pi} = 0$$

$$\Rightarrow \quad 4x + x\pi = 28\pi \Rightarrow x = \frac{28\pi}{4 + \pi}$$

$$\frac{d^2A}{dx^2} = \frac{1}{2\pi} - \frac{(-1)}{8} = \frac{1}{2\pi} + \frac{1}{8} \text{ and } \left(\frac{d^2A}{dx^2}\right)_{x=\frac{28\pi}{4+\pi}} = \frac{1}{2\pi} + \frac{1}{8} > 0$$

Hence, area A is minimum.

Hence, the length of the two pieces are

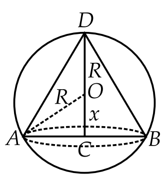
$$\frac{28\pi}{4 + \pi} \text{ m and } \left(28 - \frac{28\pi}{4 + \pi}\right) \text{m } i.e., \frac{112}{4 + \pi} \text{ m.}$$

23. Let ABD be a cone of greatest volume inscribed in the sphere.

Let $OC = x$. Then in ΔOAC, $AC = \sqrt{R^2 - x^2}$

and $DC = DO + OC = R + x$ = height of the cone.

Let V be the volume of the cone. Then,



$$V = \frac{1}{3}\pi(AC)^2(DC)$$

$$\Rightarrow \quad V = \frac{1}{3}\pi(R^2 - x^2)(R + x)$$

$$= \frac{1}{3}\pi(R^3 + xR^2 - Rx^2 - x^3) \quad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$\frac{dV}{dx} = \frac{1}{3}\pi(R^2 - 2Rx - 3x^2)$$

For maximum/minimum, we must have

$$\frac{dV}{dx} = 0$$

$\Rightarrow \quad R^2 - 2Rx - 3x^2 = 0 \Rightarrow (R - 3x)(R + x) = 0$

$\Rightarrow \quad R - 3x = 0 \Rightarrow x = \frac{R}{3} \qquad [\because R + x \neq 0]$

and $\frac{d^2V}{dx^2} = \frac{1}{3}\pi(-2R - 6x)$

$$\left(\frac{d^2V}{dx^2}\right)_{x=R/3} = -\frac{4}{3}R\pi < 0.$$

Thus, V is maximum when $x = \frac{R}{3}$.

Putting $x = \frac{R}{3}$ in $V = \frac{1}{3}\pi(R^2 - x^2)(R + x)$,

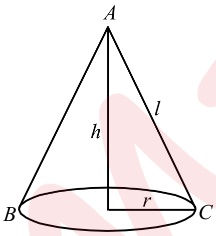
we obtain that maximum volume of the cone

$$= \frac{1}{3}\pi\left(R^2 - \frac{R^2}{9}\right)\left(R + \frac{R}{3}\right) = \frac{32\pi R^3}{81}$$

$$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right) = \frac{8}{27}\text{ (Volume of the sphere)}$$

24. Let r be the radius, h be the height and V be the volume of the cone.

Volume, $V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2}$



Surface area, $S = \pi rl$

$$= \pi r(\sqrt{h^2 + r^2}) = \pi r\left(\sqrt{\left(\frac{3V}{\pi r^2}\right)^2 + r^2}\right)$$

Squaring on both sides, we get

$$S^2 = (\pi r)^2\left[\left(\frac{3V}{\pi r^2}\right)^2 + r^2\right] = \frac{(3V)^2}{r^2} + \pi^2 r^4$$

Let $Z = S^2$

then S is maximum or minimum according as Z is maximum or minimum.

$$Z = \frac{(3V)^2}{r^2} + \pi^2 r^4 \qquad \text{...(i)}$$

Differentiate (i) w.r.t. r, we get

$$\frac{dZ}{dr} = \frac{-2(3V)^2}{r^3} + 4\pi^2 r^3$$

For maximum/minimum surface area, $\frac{dz}{dr} = 0$

$\Rightarrow \quad -2(3V)^2 + 4\pi^2 r^6 = 0 \Rightarrow -(3V)^2 + 2\pi^2 r^6 = 0$

$\Rightarrow \quad 2\pi^2 r^6 = (3V)^2$

$$\Rightarrow \quad r^6 = \frac{(3V)^2}{2\pi^2} = \frac{\left(3\left(\frac{1}{3}\pi r^2 h\right)\right)^2}{2\pi^2} = \frac{r^4 h^2}{2}$$

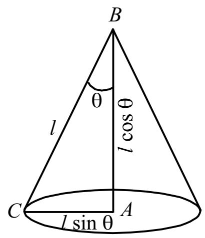
$$\Rightarrow \quad r^2 = \frac{h^2}{2} \Rightarrow 2r^2 = h^2$$

$$\frac{d^2Z}{dr^2} = -2(3V)^2\left[\frac{-3}{r^4}\right] + 12\pi^2 r^2$$

$$\left(\frac{d^2Z}{dr^2}\right)_{r^2 = \frac{h^2}{2}} > 0$$

Hence, surface area is minimum at $2r^2 = h^2$ and $h = \sqrt{2}r$.

25. If θ be the semi vertical angle, l be the given slant height, then radius of base $= l\sin\theta$, and height $= l\cos\theta$ ($\because \Delta ABC$ is a right angled triangle)



and volume of cone $(V) = \frac{1}{3}\pi r^2 h$

$$\Rightarrow \quad V = \frac{1}{3}\pi(l\sin\theta)^2 l\cos\theta$$

$$= \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta,$$

$$\Rightarrow \quad \frac{dV}{d\theta} = \frac{1}{3}\pi l^3\{(\sin^2\theta)(-\sin\theta) + \cos\theta \times 2\sin\theta\cos\theta\}$$

$$= \frac{1}{3}\pi l^3 \sin\theta[-\sin^2\theta + 2(1 - \sin^2\theta)]$$

$$= \frac{1}{3}\pi l^3 \sin\theta\cos^2\theta[2\sec^2\theta - 3\tan^2\theta]$$

$$= \frac{1}{3}\pi l^3 \sin\theta\cos^2\theta[2 - \tan^2\theta]$$

For maximum/minimum volume, $\frac{dV}{d\theta} = 0$

$$\Rightarrow \quad \frac{1}{3}\pi l^3 \sin\theta\cos^2\theta(2 - \tan^2\theta) = 0$$

$$\Rightarrow \quad \tan\theta = \sqrt{2} \Rightarrow \theta = \tan^{-1}(\sqrt{2})$$

$$\Rightarrow \quad \frac{d^2V}{d\theta^2} = \frac{1}{3}\pi l^3 \cos^3\theta(2 - 7\tan^2\theta)$$

$$\Rightarrow \quad \left(\frac{d^2V}{d\theta^2}\right)_{\tan\theta = \sqrt{2}} = \frac{1}{3}\pi l^3\left(\frac{1}{\sqrt{3}}\right)^3(2 - 7 \times 2) = -\frac{4\pi l^3}{3\sqrt{3}} < 0$$

Thus, V is maximum, when $\tan\theta = \sqrt{2}$ or $\theta = \tan^{-1}\sqrt{2}$

i.e., when the semi-vertical angle of the cone is $\tan^{-1}\sqrt{2}$.

26. Let r be the radius, l be the slant height and h be the vertical height of cone of given surface area S. Then

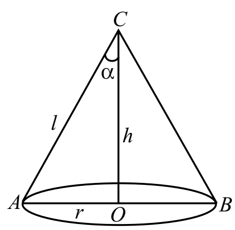
$$S = \pi rl + \pi r^2 \Rightarrow l = \frac{S - \pi r^2}{\pi r}$$

Let V be the volume of the cone. $\therefore V = \frac{1}{3}\pi r^2 h$

$$\Rightarrow \quad V^2 = \frac{1}{9}\pi^2 r^4 h^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2)$$

$$\Rightarrow \quad V^2 = \frac{\pi^2}{9} r^4\left[\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right]$$

$$= \frac{\pi^2 r^4}{9}\left[\frac{(S - \pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2}\right]$$



$$\Rightarrow \quad V^2 = \frac{1}{9}S(Sr^2 - 2\pi r^4)$$

Let $Z = V^2$, then V is maximum or minimum according as Z is maximum or minimum.

$\therefore \quad Z=\frac{1}{9}S(Sr^2-2\pi r^4) \Rightarrow \frac{dZ}{dr}=\frac{1}{9}S(2Sr-8\pi r^3)$

For maximum/minimum, we have $\frac{dZ}{dr}=0$

$\Rightarrow \quad 2Sr - 8\pi r^3 = 0 \Rightarrow S = 4\pi r^2 \Rightarrow r = \sqrt{\frac{S}{4\pi}}$

Now, $\frac{d^2Z}{dr^2}=\frac{S}{9}(2S-24\pi r^2)$

$\Rightarrow \quad \left(\frac{d^2Z}{dr^2}\right)_{r=\sqrt{\frac{S}{4\pi}}}=\frac{S}{9}\left(2S-24\pi\frac{S}{4\pi}\right)$

$\Rightarrow \quad \frac{d^2Z}{dr^2}=-\frac{4S^2}{9}<0$

So, Z is maximum, when $S = 4\pi r^2$.

Hence, V is maximum when $S = 4\pi r^2$.

Now, $S = 4\pi r^2$

$\Rightarrow \quad \pi rl + \pi r^2 = 4\pi r^2 \Rightarrow l = 3r$

$\therefore \quad \sin\alpha=\frac{r}{l}=\frac{r}{3r}=\frac{1}{3}$

Hence, V is maximum when $\alpha=\sin^{-1}\frac{1}{3}$.

27. (A) : Let Z be the distance of the point $P(x, y)$ on $x^2 = 2y$ from the point $A(0, 5)$, then $Z = |PA|^2$

$Z = (x - 0)^2 + (y - 5)^2 = 2y + (y - 5)^2 = y^2 - 8y + 25$

$\therefore \quad \frac{dZ}{dy}=2y-8$

Now, $\frac{dZ}{dy}=0 \Rightarrow 2y - 8 = 0 \Rightarrow y = 4$

$x^2 = 2(4) = 8 \quad [\because x^2 = 2y]$

$\Rightarrow \quad x=2\sqrt{2}$

$\frac{d^2Z}{dy^2}=2;\ \left(\frac{d^2Z}{dy^2}\right)_{y=4}=2>0$

Hence, Z is minimum at $(2\sqrt{2},\ 4)$.

28. (D) : Let $y=\frac{x^2-x+1}{x^2+x+1} \Rightarrow \frac{dy}{dx}=\frac{2(x-1)(x+1)}{(x^2+x+1)^2}$

For critical points, we have

$\frac{dy}{dx}=0 \Rightarrow x=1,\ -1$

At $x = 1$, $\frac{dy}{dx}$ changes from –ve to +ve.

Thus, y is minimum at $x = 1$.

$\therefore \quad$ Minimum value of $y=\frac{1-x+x^2}{1+x+x^2}$ at $x = 1$ is

$\frac{1-1+1}{1+1+1}=\frac{1}{3}$

29. (C) : Let $f(x) = [x(x - 1) + 1]^{1/3}$

$= (x^2 - x + 1)^{1/3}, 0 \le x \le 1$

$\Rightarrow \quad f'(x)=\frac{1}{3}(x^2-x+1)^{\frac{1}{3}-1}(2x-1)$

For critical points, $f'(x) = 0 \Rightarrow 2x - 1 = 0$

$\Rightarrow \quad x=\frac{1}{2}\in[0,\ 1]$.

For maximum and minimum value, we should evaluate $f(0)$, $f(1)$ and $f(1/2)$.

Here, $f(0) = (0 + 1)^{1/3} = 1$,

$f(1) = (1(0) + 1)^{1/3} = 1$

and $f\left(\frac{1}{2}\right)=\left(\frac{1}{4}-\frac{1}{2}+1\right)^{1/3}=\left(\frac{3}{4}\right)^{1/3}$

$\therefore \quad$ Maximum value of $f(x) = 1$ at $x = 0$ and 1.

NCERT MISCELLANEOUS EXERCISE

1. We have, $f(x)=\frac{\log x}{x},\ x>0$ ...(i)

Differentiating (i) w.r.t. x, we get

$\Rightarrow \quad f'(x)=\frac{x\left(\frac{1}{x}\right)-(\log x)\cdot 1}{x^2}=\frac{1-\log x}{x^2}$ ...(ii)

For maximum/minimum, $f'(x) = 0$

$\Rightarrow \quad \frac{1-\log x}{x^2}=0 \Rightarrow \log x=1 \quad (\because x^2\neq 0)$

$\Rightarrow \quad x = e$

Again differentiating (ii) w.r.t. x, we get

$f''(x)=\frac{x^2\left(-\frac{1}{x}\right)-(1-\log x)2x}{x^4}=\frac{-x-2x+2x\log x}{x^4}$

$=\frac{x(2\log x-3)}{x^4}=\frac{2\log x-3}{x^3}$.

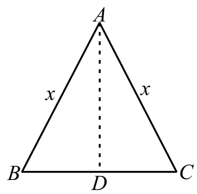
Also, $f''(e)=\frac{2\log e-3}{e^3}=\frac{2\cdot 1-3}{e^3} \quad [\because \log_e e=1]$

$=\frac{-1}{e^3}<0$

$\Rightarrow \quad f(x)$ has a maximum value at $x = e$.

2. Let us take either of the equal sides AB and AC be x

$\Rightarrow \quad \frac{dx}{dt}=-3$ cm/sec.



Let A is the corresponding area of ΔABC

$\therefore \quad A=\frac{1}{2}\text{base}\times\text{height}=\frac{1}{2}b\sqrt{AB^2-BD^2}$

$=\frac{b}{2}\sqrt{x^2-\left(\frac{b}{2}\right)^2}=\frac{b}{4}\sqrt{4x^2-b^2}$

$$\frac{dA}{dt} = \frac{b}{4}\times\frac{1}{2}\times\frac{8x}{\sqrt{4x^2-b^2}}\,\frac{dx}{dt} = \frac{bx}{\sqrt{4x^2-b^2}}\,\frac{dx}{dt}$$

$$\left(\frac{dA}{dt}\right)_{x=b} = \left(\frac{b\times b}{\sqrt{4b^2-b^2}}\right)(-3) = -\sqrt{3}b.$$

$\Rightarrow$ Area is decreasing at the rate of $b\sqrt{3}\,\text{cm}^2/\text{sec}$.

3. We have, $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2+\cos x}$

$$= \frac{4\sin x - x(2+\cos x)}{2+\cos x}$$

$$= \frac{4\sin x}{2+\cos x} - \frac{x(2+\cos x)}{2+\cos x} = \frac{4\sin x}{2+\cos x} - x \qquad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$\Rightarrow \quad f'(x) = 4\left\{\frac{(2+\cos x)\cos x - \sin x(0-\sin x)}{(2+\cos x)^2}\right\} - 1$$

$$= 4\left\{\frac{2\cos x + \cos^2 x + \sin^2 x}{(2+\cos x)^2}\right\} - 1$$

$$= \frac{8\cos x + 4 - (2+\cos x)^2}{(2+\cos x)^2} = \frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

Now $(2 + \cos x)^2$, being a perfect square, is always non-negative.

$(2 + \cos x)^2 \neq 0$ as $(2 + \cos x) \neq 0$ [$\because \cos x \neq -2$]

$\therefore$ $(2 + \cos x)^2$ is always +ve and $4 - \cos x$ is also always +ve, as $\cos x$ is always numerically ≤ 1

$\therefore$ $f'(x)$ is +ve or –ve according as

$$0 < x < \frac{\pi}{2} \quad \text{and} \quad \frac{3\pi}{2} < x < 2\pi \quad \text{or} \quad \frac{\pi}{2} < x < \frac{3\pi}{2}$$

Hence, $f(x)$ is an increasing function on $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

and decreasing function on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

4. We have, $f(x) = x^3 + \dfrac{1}{x^3}$ ...(i)

Differentiating (i) w.r.t. x, we get

$$f'(x) = 3x^2 - \frac{3}{x^4}$$

For $f(x)$ to be increasing function of x,

$$f'(x) > 0 \quad \Rightarrow \quad 3\left(x^2 - \frac{1}{x^4}\right) > 0$$

$\Rightarrow$ $x^6 - 1 > 0$ $\Rightarrow$ $(x^3 - 1)(x^3 + 1) > 0$

Either $x^3 - 1 > 0$ and $x^3 + 1 > 0$

$\Rightarrow$ $x^3 > 1$ or $x^3 > -1$ $\Rightarrow$ $x > 1$ and $x > -1$

$\Rightarrow$ $x > 1$ $\Rightarrow$ $x \in (1, \infty)$

or $x^3 - 1 < 0$ and $x^3 + 1 < 0$

$\Rightarrow$ $x^3 < 1$ and $x^3 < -1$ $\Rightarrow$ $x < 1$ and $x < -1$

$\Rightarrow$ $x < -1$ $\Rightarrow$ $x \in (-\infty, -1)$

Hence, $f(x)$ is increasing in $(-\infty, -1) \cup (1, \infty)$.

For $f(x)$ to be decreasing function of x,

$$f'(x) < 0 \Rightarrow 3\left(x^2 - \frac{1}{x^4}\right) < 0$$

$$\Rightarrow \quad x^2 - \frac{1}{x^4} < 0 \;\Rightarrow\; x^6 - 1 < 0 \;\Rightarrow\; (x^3 - 1)(x^3 + 1) < 0$$

Either $x^3 - 1 > 0$ and $x^3 + 1 < 0$

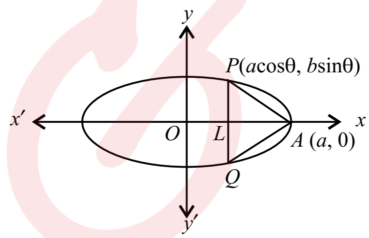
$\Rightarrow$ $x^3 > 1$ and $x^3 < -1$ $\Rightarrow$ $x > 1$ and $x < -1$, which is not possible

or $x^3 - 1 < 0$ and $x^3 + 1 > 0$

$\Rightarrow$ $x^3 < 1$ and $x^3 > -1$ $\Rightarrow$ $x < 1$ and $x > -1$ $\Rightarrow -1 < x < 1$.

Hence, $f(x)$ is decreasing in $(-1, 1)$.

5. Let us take that the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then any point P on the ellipse is $(a\cos\theta, b\sin\theta)$.



From P, draw PQ parallel to y-axis and produce it to meet the ellipse at Q, then PAQ is an isosceles triangle, let A be its area, then

$$A = \frac{1}{2}PQ\cdot AL = \frac{1}{2}(2b\sin\theta)(a - a\cos\theta)$$

$A = (a - a\cos\theta) \times b\sin\theta$

$\Rightarrow$ $A = ab(\sin\theta - \sin\theta\cos\theta)$ [$\because AL = OA - OL$]

$$\Rightarrow \quad A = ab\left(\sin\theta - \frac{1}{2}\sin 2\theta\right) \qquad \text{...(i)}$$

Differentiating (i) w.r.t. θ, we get

$$\frac{dA}{d\theta} = ab(\cos\theta - \cos 2\theta) \qquad \text{...(ii)}$$

For maxima/minima,

$$\frac{dA}{d\theta} = 0 \;\Rightarrow\; \cos\theta = \cos 2\theta$$

$\cos 2\theta = \cos(2\pi - \theta)$

$$\Rightarrow \quad 2\theta = 2\pi - \theta \;\Rightarrow\; \theta = \frac{2\pi}{3}$$

Differentiating (ii) w.r.t. θ, we get

$$\frac{d^2A}{d\theta^2} = ab(-\sin\theta + 2\sin 2\theta)$$

$$\left(\frac{d^2A}{d\theta^2}\right)_{\theta=2\pi/3} = ab\left(-\sin\frac{2\pi}{3} + 2\sin\frac{4\pi}{3}\right)$$

$$= ab\left(-\frac{\sqrt{3}}{2} - \sqrt{3}\right) = \frac{-3\sqrt{3}}{2}ab < 0.$$

$\therefore$ A is maximum when $\theta = \dfrac{2\pi}{3}$

Also, the maximum area is

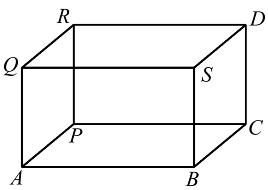
$$A = ab\left(\sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3}\right) = ab\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$= \frac{3\sqrt{3}}{4} ab$ sq. units.

6. Let the dimensions of the base be x metres and y metres. The total expenses be ₹ S and the volume of the tank be V m^3, then $V = 2xy$ but $V = 8$(given)

$\Rightarrow \quad 2xy = 8 \quad \Rightarrow \quad y = \frac{4}{x}$ ...(i)

Also, area of the base $ABCP = xy = x \times \frac{4}{x} = 4$ m^2 (From (i))



and area of the four sides ($ABSQ$, $PCDR$, $BCDS$, $APRQ$) $= (2x + 2x + 2y + 2y)$ m^2

$= \left(4x + 4\left(\frac{4}{x}\right)\right)$ m^2 (From (i))

$= 4\left(x + \frac{4}{x}\right)$ m^2

$\therefore \quad S = 4 \times 70 + 45 \times 4\left(x + \frac{4}{x}\right) = 280 + 180\left(x + \frac{4}{x}\right)$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$\frac{dS}{dx} = 180\left(1 - \frac{4}{x^2}\right)$ ...(iii)

For maximum/minimum expenses, $\frac{dS}{dx} = 0$

$\Rightarrow \quad 180\left(1 - \frac{4}{x^2}\right) = 0 \quad \Rightarrow x^2 = 4 \Rightarrow x = 2$

($\because$ x can't be negative)

Differentiating (iii) w.r.t. x, we get

$\frac{d^2S}{dx^2} = 180\left(0 + \frac{8}{x^3}\right) = \frac{1440}{x^3}$

$\left(\frac{d^2S}{dx^2}\right)_{x=2} = \frac{1440}{2^3} > 0$

$\therefore$ S is least when $x = 2$

$\therefore$ Cost of least expensive tank = ₹ S

$=$ ₹ $\left\{280 + 180\left(2 + \frac{4}{2}\right)\right\}$ (From (ii))

= ₹ (280 + 720) = ₹ 1000.

7. Let r be the radius of the circle and x be the side of the square, then

$2\pi r + 4x = k$ ...(i)

Let S be the sum of areas of the circle and the square, then

$S = \pi r^2 + x^2 = \pi r^2 + \left(\frac{k - 2\pi r}{4}\right)^2$ (Using (i))

$= \pi r^2 + \frac{k^2}{16} + \frac{\pi^2 r^2}{4} - \frac{k\pi r}{4}$ ...(ii)

Differentiating (ii) w.r.t. r, we get

$\frac{dS}{dr} = 2\pi r + \frac{2\pi^2 r}{4} - \frac{k\pi}{4}$ ...(iii)

For maximum/minimum, $\frac{dS}{dr} = 0$

$\Rightarrow \quad 2\pi r + \frac{2\pi^2 r}{4} - \frac{k\pi}{4} = 0 \Rightarrow r\left(2\pi + \frac{\pi^2}{2}\right) = \frac{k\pi}{4}$

$\Rightarrow \quad r = \frac{2k\pi}{4(4\pi + \pi^2)} = \frac{k}{2(4 + \pi)}$.

Differentiating (iii) w.r.t. r, we get

$\frac{d^2S}{dr^2} = 2\pi + \frac{\pi^2}{2}$

$\left(\frac{d^2S}{dr^2}\right)_{r = \frac{k}{2(4+\pi)}} = 2\pi + \frac{\pi^2}{2} > 0$

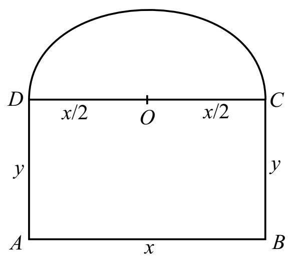
$\Rightarrow$ S is minimum at $r = \frac{k}{2(4 + \pi)}$

$\Rightarrow \quad x = \frac{1}{4}\left\{k - \frac{2\pi}{2}\left(\frac{k}{4 + \pi}\right)\right\}$ (From (i))

$= \frac{4k}{4(4 + \pi)} = 2\left[\frac{k}{2(4 + \pi)}\right] = 2(r)$

Hence, S is least when side of the square is double the radius of the circle.

8. Let x and y be the length and breadth of the rectangle.



Radius of the semi-circle $= \frac{x}{2}$

Circumference of the semi-circle $= \frac{\pi x}{2}$.

Perimeter of the window = $AB + BC + AD + \widehat{DC}$

$x + 2y + \frac{\pi x}{2} = 10$ (Given)

$\Rightarrow \quad 2x + 4y + \pi x = 20$

$\Rightarrow \quad y = \frac{20 - (2 + \pi)x}{4}$ ...(i)

Area of the window = area of rectangle + area of semi-circle

$A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = x\left(\frac{20 - (2 + \pi)x}{4}\right) + \frac{\pi x^2}{8}$. (Using (i))

$A = \frac{20x - (2 + \pi)x^2}{4} + \frac{\pi x^2}{8}$.

$\therefore \quad \frac{dA}{dx} = \frac{20 - (2 + \pi)2x}{4} + \frac{2\pi x}{8}$

For maxima/minima of A,

$\frac{dA}{dx} = 0 \Rightarrow \frac{20-(2+\pi)2x}{4} + \frac{2\pi x}{8} = 0$

$\Rightarrow$ $20 - (2 + \pi)2x + \pi x = 0$

$\Rightarrow$ $20 + x(\pi - 4 - 2\pi) = 0$

$\Rightarrow$ $20 - x(4+\pi) = 0 \Rightarrow x = \frac{20}{4+\pi}$

$$\frac{d^2A}{dx^2} = \frac{-(2+\pi)2}{4} + \frac{2\pi}{8} = \frac{-4-2\pi+\pi}{4} = \frac{-4-\pi}{4}$$

$$\Rightarrow \left(\frac{d^2A}{dx^2}\right)_{x=\frac{20}{4+\pi}} = -\left(\frac{4+\pi}{4}\right) < 0$$

Hence, the window admit the maximum light,

when length is $x = \frac{20}{4+\pi}$

and breadth, $y = \dfrac{20-(2+\pi)\frac{20}{4+\pi}}{4}$

$$= \frac{80+20\pi-40-20\pi}{4(4+\pi)} = \frac{40}{4(4+\pi)} = \frac{10}{4+\pi}.$$

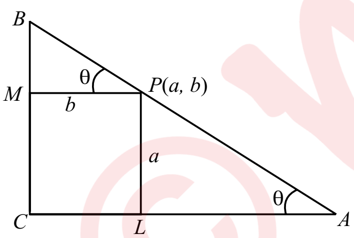
9. Let ABC be a right angled triangle with hypotenuse AB.

Let P be on AB such that $PL = a$ and $PM = b$. Let $\angle CAB = \theta$.

$\therefore$ $AP = a \operatorname{cosec} \theta$ and $BP = b \sec\theta$.

If l be the length of the hypotenuse AB, then, $l = AP + BP = a \operatorname{cosec} \theta + b \sec\theta$.

$\therefore$ $\frac{dl}{d\theta} = -a\operatorname{cosec}\theta\cot\theta + b\sec\theta\tan\theta$



For maximum/minimum, $\frac{dl}{d\theta} = 0$

$\Rightarrow$ $-a \operatorname{cosec}\theta \cot\theta + b \sec\theta \tan\theta = 0$

$\Rightarrow$ $-\frac{a\cos\theta}{\sin^2\theta} + \frac{b\sin\theta}{\cos^2\theta} = 0$

$\Rightarrow$ $\frac{a\cos\theta}{\sin^2\theta} = \frac{b\sin\theta}{\cos^2\theta} \Rightarrow \tan^3\theta = \frac{a}{b}$

$\Rightarrow$ $\tan\theta = \left(\frac{a}{b}\right)^{1/3}$

$\therefore$ $\sin\theta = \frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}, \cos\theta = \frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$

and $\frac{d^2l}{d\theta^2} = a\operatorname{cosec}^3\theta + a\operatorname{cosec}\theta\cot^2\theta + b\sec^3\theta + b\sec\theta\tan^2\theta$

and $\frac{d^2l}{d\theta^2} > 0$ when $\tan\theta = \left(\frac{a}{b}\right)^{1/3}$

Thus, l is minimum when $\tan\theta = \left(\frac{a}{b}\right)^{1/3}$

$\therefore$ Minimum value of $l = a\sqrt{1+\cot^2\theta} + b\sqrt{1+\tan^2\theta}$

$$= a\sqrt{1+\left(\frac{b}{a}\right)^{2/3}} + b\sqrt{1+\left(\frac{a}{b}\right)^{2/3}}$$

$$= a^{2/3}\sqrt{a^{2/3}+b^{2/3}} + b^{2/3}\sqrt{b^{2/3}+a^{2/3}}$$

$= (a^{2/3} + b^{2/3})^{3/2}$.

10. We have, $f(x) = (x - 2)^4(x + 1)^3$.

$\therefore$ $f'(x) = (x - 2)^4 \cdot 3(x + 1)^2 + (x + 1)^3 \cdot 4(x - 2)^3$

$= (x - 2)^3 (x + 1)^2 [3(x - 2) + 4(x + 1)]$

$= (x - 2)^3 (x + 1)^2 (7x - 2) = 7(x - 2)^3 (x + 1)^2 \left(x - \frac{2}{7}\right)$.

For maximum/minimum, $f'(x) = 0$

$\Rightarrow$ $7(x-2)^3(x+1)^2\left(x-\frac{2}{7}\right) = 0$

$\Rightarrow$ $x = 2, -1, \frac{2}{7}$.

At $x = 2$: When $x < 2$ slightly, $f'(x) = (-) (+) (+) = -$ve

When $x > 2$ slightly, $f'(x) = (+) (+) (+) = +$ve

$\therefore$ $f'(x)$ changes from –ve to +ve while passing through the point $x = 2$.

Thus, $f(x)$ is minimum at $x = 2$

At $x = -1$: When $x < -1$ slightly, $f'(x) = (-) (+) (-) = +$ve

When $x > -1$ slightly, $f'(x) = (-) (+) (-) = +$ve

$\therefore$ $f'(x)$ does not change its sign while passing through the point $x = -1$.

Thus, $x = -1$ is a point of inflexion.

At $x = \frac{2}{7}$: When $x < \frac{2}{7}$ slightly, $f'(x) = (-) (+) (-) = +$ve

When $x > \frac{2}{7}$ slightly, $f'(x) = (-) (+) (+) = -$ve

$\therefore$ $f'(x)$ changes from +ve to –ve while passing through the point $x = \frac{2}{7}$.

Thus, $f(x)$ is maximum at $x = \frac{2}{7}$.

11. We have, $f(x) = \cos^2 x + \sin x, x \in [0, \pi]$

$\therefore$ $f'(x) = 2\cos x(-\sin x) + \cos x$

$= \cos x(-2\sin x + 1)$

For maximum/minimum values, $f'(x) = 0$

$\Rightarrow$ $\cos x(-2\sin x + 1) = 0$

$\Rightarrow$ $\cos x = 0$ or $-2\sin x + 1 = 0$

$\Rightarrow$ $x = \frac{\pi}{2}$ or $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$.

For maximum and minimum value, we evaluate $f(0)$, $f(\pi)$, $f(\pi/2)$ and $f(\pi/6)$.

Now, $f(0) = \cos^2 0 + \sin 0 = (1)^2 + 0 = 1$

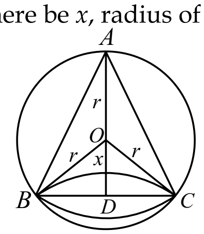
$$f\left(\frac{\pi}{6}\right) = \cos^2\frac{\pi}{6} + \sin\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$f\left(\frac{\pi}{2}\right) = \cos^2\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$ and

$f(\pi) = \cos^2\pi + \sin\pi = (-1)^2 + 0 = 1.$

Hence, absolute maximum and minimum values are $\frac{5}{4}$ and 1 respectively.

12. Let the distance of the base BC of the cone from the centre O of the sphere be x, radius of the sphere is r.



Height of the cone, $H = AD = r + x$

and radius of the base of cone,

$$R = BD = \sqrt{r^2 - x^2}.$$

Let V be the volume of the cone,

$$\therefore \quad V = \frac{1}{3}\pi R^2 H = \frac{1}{3}\pi(r^2 - x^2)(r + x)$$

$$\therefore \quad \frac{dV}{dx} = \frac{\pi}{3}[(r^2 - x^2)(0+1) + (r+x)(0-2x)]$$

$$= \frac{\pi}{3}[(r-x)(r+x) - 2x(r+x)]$$

$$= \frac{\pi}{3}[(r+x)(r-x-2x)] = \frac{\pi}{3}(r+x)(r-3x).$$

For maximum/minimum,

$$\frac{dV}{dx} = 0 \Rightarrow \frac{\pi}{3}(r+x)(r-3x) = 0$$

$$\Rightarrow \quad (r+x)(r-3x) = 0 \Rightarrow x = -r, \frac{r}{3}.$$

But $x \neq -r$, $\therefore$ $x = \frac{r}{3}$.

$$\text{Now} \quad \frac{d^2V}{dx^2} = \frac{\pi}{3}[(r+x)(-3) + (r-3x)(1)]$$

$$= \frac{\pi}{3}(-3r - 3x + r - 3x) = \frac{\pi}{3}(-2r - 6x)$$

$$\therefore \quad \left[\frac{d^2V}{dx^2}\right]_{x=\frac{r}{3}} = \frac{\pi}{3}\left[-2r - 6\left(\frac{r}{3}\right)\right] = \frac{\pi}{3}(-2r - 2r)$$

$$= -\frac{4r\pi}{3} < 0$$

Hence, V is maximum when $x = \frac{r}{3}$ and

the altitude $AD = r + x = r + \frac{r}{3} = \frac{4r}{3}.$

13. Let $x_1, x_2 \in (a, b) \Rightarrow x_1 < x_2$

Let the sub-interval be $[x_1, x_2]$.

Since, $f(x)$ is differentiable on (a, b) and $[x_1, x_2] \subset (a, b)$

$\therefore$ $f(x)$ is continuous on $[x_1, x_2]$ and differentiable in (x_1, x_2).

$\therefore$ By LMV theorem, there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Now, $f'(x) > 0$ for all $x \in (a, b) \Rightarrow f'(c) > 0$

$$\Rightarrow \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \Rightarrow f(x_2) - f(x_1) > 0$$

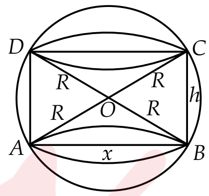
$[\because x_1 < x_2 \Rightarrow x_2 - x_1 > 0]$

$\Rightarrow \quad f(x_1) < f(x_2)$ if $x_1 < x_2$

Hence, f is an increasing in (a, b). $[\because x_1, x_2$ are arbitrary$]$

14. Let h be the height and x be the diameter of the base of the inscribed cylinder.

Then in ΔABC, $h^2 + x^2 = 4R^2$ ...(i)



Now, V be the volume of the cylinder

$$V = \pi\left(\frac{x}{2}\right)^2 h = \frac{1}{4}\pi x^2 h = \frac{1}{4}\pi(4R^2 - h^2)h \quad \text{(From (i))}$$

$$= \pi R^2 h - \frac{1}{4}\pi h^3.$$

$$\text{Now,} \quad \frac{dV}{dh} = \pi R^2 - \frac{3}{4}\pi h^2 = \pi\left(R^2 - \frac{3}{4}h^2\right).$$

For maxima/minima, $\frac{dV}{dh} = 0$

$$\Rightarrow \quad \pi\left(R^2 - \frac{3}{4}h^2\right) = 0 \Rightarrow R^2 = \frac{3}{4}h^2 \quad \Rightarrow \quad h = \frac{2R}{\sqrt{3}}.$$

$$\text{and} \quad \frac{d^2V}{dh^2} = -\frac{3}{4}\pi(2h) = -\frac{3}{2}\pi h;$$

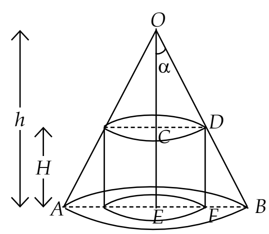
$$\left(\frac{d^2V}{dh^2}\right)_{h=\frac{2R}{\sqrt{3}}} = -\frac{3}{2}\pi\left(\frac{2R}{\sqrt{3}}\right) < 0$$

Thus, V is maximum at $h = \frac{2R}{\sqrt{3}}$ and

$$\text{maximum volume} = \frac{1}{4}\pi\left(\frac{2R}{\sqrt{3}}\right)\left(4R^2 - \frac{4R^2}{3}\right)$$

$$= \frac{\pi R}{2\sqrt{3}}\left(\frac{8R^2}{3}\right) = \frac{4\pi R^3}{3\sqrt{3}} \text{ cubic units.}$$

15. Let R be the radius, H be the height of the cylinder inscribed in cone, r be the radius and h be the height of the cone.



Then, $OC = OE - CE = h - H$ and $CD = R$.

$$\text{Now in } \Delta OCD, \ \tan\alpha = \frac{CD}{OC} = \frac{R}{h - H}$$

$\Rightarrow \quad R = (h - H)\tan\alpha,$ ...(i) where α is the semi-vertical angle of the cone

Let V be the volume of the cylinder

$\Rightarrow \quad V = \pi R^2 H = \pi[(h - H)^2 \tan^2\alpha]H$ (Using (i))

$= \pi H \cdot (h - H)^2 \tan^2\alpha$ ...(ii)

Differentiating w.r.t. H, we get

$$\frac{dV}{dH} = \pi[(h-H)^2 \times 1 + H \cdot 2(h-H)(-1)]\tan^2\alpha$$

$= \pi \tan^2\alpha(h^2 - 4hH + 3H^2)$ ...(iii)

$= \pi\tan^2\alpha(h - H)(h - 3H)$

Now, $\frac{dV}{dH} = 0 \Rightarrow (h-H)(h-3H) = 0$

$\Rightarrow \quad h = H, 3H \Rightarrow H = h$ or $\frac{h}{3}$.

($\because$ Cylinder is inscribed in the cone)

$\Rightarrow \quad H \neq h$. Hence, $H = \frac{h}{3}$.

Differentiating (iii) w.r.t. H, we get

$$\frac{d^2V}{dH^2} = \pi \tan^2\alpha(0 - 4h + 6H)$$

$$\left(\frac{d^2V}{dH^2}\right)_{H=\frac{h}{3}} = \pi\tan^2\alpha\left(-4h + 6\left(\frac{h}{3}\right)\right) = \pi\tan^2\alpha(-2h) < 0$$

$\Rightarrow \quad \frac{d^2V}{dH^2} < 0$ at $H = \frac{h}{3}$

Hence, the volume of the inscribed cylinder is maximum when its height is $\frac{h}{3}$.

Radius of the cylinder = CD = $OC \tan\alpha = (h - H)\tan\alpha$

$= \left(h - \frac{h}{3}\right)\tan\alpha = \frac{2}{3}h\tan\alpha$

$\therefore$ Volume of the cylinder $= \pi\left(\frac{2}{3}h\tan\alpha\right)^2\left(\frac{h}{3}\right)$

$= \frac{4}{27}\pi h^3 \tan^2\alpha.$

16. (A) : Let h be the depth of the cylindrical tank.

$\therefore$ V be the volume of cylindrical tank

$\Rightarrow \quad V = \pi r^2 h = \pi(10)^2 h = 100\pi h$

Differentiating with respect to t, we get

$\therefore$ Rate of change of volume, $\frac{dV}{dt} = 100\pi\frac{dh}{dt}$

$314 = 100\pi\frac{dh}{dt} \qquad \left(\frac{dV}{dt} = 314 \text{ (given)}\right)$

$\Rightarrow \quad \frac{dh}{dt} = \frac{314}{100\pi} \approx \frac{314}{100 \times 3.14} = \frac{314}{314} = 1\text{ m/h}.$