CHAPTER 6

Application of Derivatives

TRY YOURSELF SOLUTIONS

1. Let A be the area of the circle and r be its radius. Then, $A = \pi r^2$.

$$\Rightarrow \quad \frac{dA}{dr} = 2\pi r$$

Thus, the rate of change of the area of the circle with respect to its radius r is $\frac{dA}{dr} = 2\pi r$.

When $r = 5$ cm. $\frac{dA}{dr} = 2\pi \times 5 = 10\,\pi$ cm.

Thus, the area of the circle is changing at the rate of 10π cm.

2. Diameter of balloon $= \frac{2}{3} \times (x+8)$ unit

$\therefore$ Radius of balloon $= \frac{1}{3} \times (x+8)$ unit

Let V be the volume of the balloon. Then,

$$V = \frac{4}{3}\pi\left\{\frac{1}{3}(x+8)\right\}^3 = \frac{4}{81}\pi(x+8)^3$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{4}{81}\pi \times 3(x+8)^2 = \frac{4\pi}{27}(x+8)^2 \text{ unit}^2$$

3. Let 'a' be the side of an equilateral triangle.

Then, $\frac{da}{dt} = 2\,\text{cm/sec}$

Let 'A' be the area of the equilateral triangle, then

$$A = \frac{\sqrt{3}}{4}a^2 \quad \Rightarrow \quad \frac{dA}{dt} = 2 \times \frac{\sqrt{3}}{4} a \frac{da}{dt} = \frac{\sqrt{3}}{2} a \frac{da}{dt}$$

$$\therefore \quad \left(\frac{dA}{dt}\right)_{a=20} = \frac{\sqrt{3}}{2} \times 20 \times 2 = 20\sqrt{3}\,\text{cm}^2/\text{sec}$$

4. Let $x_1, x_2 \in R$ be such that $x_1 < x_2$. Then,

$x_1 < x_2 \Rightarrow -3x_1 > -3x_2 \Rightarrow -3x_1 + 12 > -3x_2 + 12$

$\Rightarrow f(x_1) > f(x_2)$

Thus, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in R$.

So, $f(x)$ is strictly decreasing function on R.

5. We have, $f(x) = x^3 - 8 \Rightarrow f'(x) = 3x^2 \Rightarrow f'(x) > 0$ for all $x \in [1, 2]$

So, $f(x)$ is increasing on $[1, 2]$.

6. Here, $f(x) = x^2 - 4x + 6$

$\Rightarrow f'(x) = 2x - 4 = 2(x - 2)$

For $f(x)$ to be increasing, we must have $f'(x) > 0$

$\Rightarrow 2(x - 2) > 0 \Rightarrow x > 2 \Rightarrow x \in (2, \infty)$

For $f(x)$ to be decreasing, we must have $f'(x) < 0$

$\Rightarrow 2(x - 2) < 0 \Rightarrow x < 2 \Rightarrow x \in (-\infty, 2)$

So, $f(x)$ is decreasing on $(-\infty, 2)$.

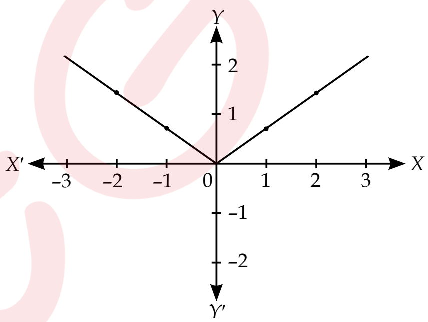
7. Here $f(x) = x + \sin x \Rightarrow f'(x) = 1 + \cos x$

Now, least value of $\cos x = -1$ at $x = \pi$.

$\therefore f'(x) = 1 + \cos x \geq 0$ for all x.

Hence, $f(x)$ is an increasing function for all x.

8. We have, $f(x) = |x|$ for all $x \in R$.



Clearly, $|x| \geq 0$ for all $x \in R$

$\Rightarrow f(x) \geq 0 \ \forall x \in R$.

So, minimum value of $f(x)$ is 0 at $x = 0$. Also, graph clearly shows that f has no maximum value in R and hence, no point of maximum value in R.

9. Let $f(x) = \alpha$

$\therefore f'(x) = 0$ for all x [$\because \alpha$ is constant]

Let 'a' be any real number, then $f'(a) = 0$

When x is slightly $< a$, $f'(x) = 0$

When x is slightly $> a$, $f'(x) = 0$

$\therefore f'(x)$ does not change sign. Thus, a is neither a point of local maximum nor a point of local minimum. Hence, $f(x)$ has neither local maximum nor local minimum.

10. Let $f(x) = x^3 - 3x + 3$

$\therefore f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$

Now, $f'(x) = 0 \Rightarrow x = -1$ or $x = 1$

Now, we will test the nature of the function at the point $x = -1, 1$

At $x = -1$:

When x is slightly < -1, $f'(x) = (-)(-) = +$ve

When x is slightly > -1, $f'(x) = (-)(+) = -$ve

Thus, $f'(x)$ changes sign from positive to negative as x increases through -1.

$\therefore f(x)$ has a local maximum at $x = -1$ and local maximum value is $f(-1) = (-1)^3 - 3(-1) + 3 = -1 + 3 + 3 = 5$

At $x = 1$:

When x is slightly < 1, $f'(x) = (-)(+) = -$ve

When x is slightly > 1, $f'(x) = (+)(+) = +$ve

Thus, $f'(x)$ changes sign from negative to positive as x increases through 1.

$\therefore$ $f(x)$ has a local minimum at $x = 1$

Local minimum value $f(1) = (1)^3 - 3(1) + 3 = 1$

11. Here, $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$

$\Rightarrow$ $f'(x) = 12x^3 + 12x^2 - 24x$

For local maximum or local minimum, $f'(x) = 0$

$\Rightarrow$ $12(x^3 + x^2 - 2x) = 0 \Rightarrow 12x(x^2 + x - 2) = 0$

$\Rightarrow$ $12x(x-1)(x+2) = 0 \Rightarrow x = 0, 1, -2$

Now, $f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$

At $x = 0 : f''(0) = -24 < 0$

$\Rightarrow$ $f(x)$ has a local maximum at $x = 0$

At $x = 1 : f''(1) = 12 \times 3 = 36 > 0$

$\Rightarrow$ $f(x)$ has local minimum at $x = 1$

At $x = -2 : f''(-2) = 12(12 - 4 - 2) = 72 > 0$

$\Rightarrow$ $f(x)$ has local minimum at $x = -2$

Local maximum value is $f(0) = 12$

Local minimum value is $f(1) = 3 + 4 - 12 + 12 = 7$

and $f(-2) = 3 \times 16 - 4 \times 8 - 12 \times 4 + 12 = -20$

12. We have, $f(x) = 2\cos x + x$, where $0 < x < \pi$.

$\Rightarrow$ $f'(x) = -2\sin x + 1$

For local maximum and minimum, $f'(x) = 0$

$\Rightarrow$ $-2\sin x + 1 = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$[\because 0 < x < \pi]$

Thus, $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ are possible points of local maximum or minimum.

Now, we test the function at these points.

Clearly, $f''(x) = -2\cos x$

At $x = \pi/6$: We have, $f''\left(\frac{\pi}{6}\right) = -2\cos\frac{\pi}{6} = -\sqrt{3} < 0$

So, $x = \frac{\pi}{6}$ is a point of local maximum.

The local maximum value of $f(x)$ is

$$f\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6} + \frac{\pi}{6} = \sqrt{3} + \frac{\pi}{6}$$

At $x = 5\pi/6$: We have, $f''\left(\frac{5\pi}{6}\right) = -2\cos\frac{5\pi}{6} = \sqrt{3} > 0$

So, $x = \frac{5\pi}{6}$ is a point of local minimum.

The local minimum value of $f(x)$ is

$$f\left(\frac{5\pi}{6}\right) = 2\cos\frac{5\pi}{6} + \frac{5\pi}{6} = -\sqrt{3} + \frac{5\pi}{6}.$$

13. Let $f(x) = 1 - x - 2x^2$

$\Rightarrow$ $f'(x) = -1 - 4x = -(1 + 4x)$

For extreme values, put $f'(x) = 0 \Rightarrow -(1 + 4x) = 0$

$\Rightarrow$ $x = \frac{-1}{4}$

Now, we find the values, of $f(x)$ at $x = -1, \frac{-1}{4}, 1$

$f(-1) = 1 - (-1) - 2(-1)^2 = 1 + 1 - 2 = 0,$

$$f\left(\frac{-1}{4}\right) = 1 - \left(-\frac{1}{4}\right) - 2\left(-\frac{1}{4}\right)^2 = 1 + \frac{1}{4} - \frac{1}{8} = \frac{8+2-1}{8} = \frac{9}{8}$$

$f(1) = 1 - 1 - 2(1)^2 = -2$

$\therefore$ Absolute maximum value $= \frac{9}{8}$ at $x = \frac{-1}{4}$ and

Absolute minimum value $= -2$ at $x = 1$

14. Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 1$. Then,

$f'(x) = 12x^3 - 24x^2 + 24x - 48$ and $f''(x) = 36x^2 - 48x + 24$

The critical points of $f(x)$ are given by $f'(x) = 0$.

$\Rightarrow$ $12x^3 - 24x^2 + 24x - 48 = 0$

$\Rightarrow$ $x^3 - 2x^2 + 2x - 4 = 0 \Rightarrow x^2(x-2) + 2(x-2) = 0$

$\Rightarrow$ $(x-2)(x^2+2) = 0 \Rightarrow x = 2$ $\quad [\because x^2 + 2 \neq 0]$

The values of $f(x)$ at critical points and at the end-points of the interval are computed as follows :

$f(2) = -63, f(1) = -40$ and $f(4) = 257$.

Of these values the largest and the smallest values are $f(4) = 257$ and $f(2) = -63$.

So, the minimum and maximum values of $f(x)$ on [1, 4] are -63 and 257 respectively.

15. Let one part be x.

Then, second part $= 30 - x$

Let the product $= p$

$\therefore$ $p = x(30 - x) = 30x - x^2$

$\therefore$ $\frac{dp}{dx} = 30 - 2x$

Now, for p to be maximum or minimum, $\frac{dp}{dx} = 0$ *i.e.*, $30 - 2x = 0 \Rightarrow x = 15$

Also, $\frac{d^2p}{dx^2} = -2$ (–ve) $\Rightarrow p$ is maximum when $x = 15$.

$\therefore$ Product is maximum when first part = 15 and second part = 30 – 15 = 15.

16. Let x and y be the length and breadth of the rectangle whose perimeter (P) is given as

$$P = 2(x + y) \Rightarrow y = \frac{P}{2} - x$$

Now, area of rectangle $(A) = xy$

$\Rightarrow$ $A = x\left(\frac{P}{2} - x\right)$

$\Rightarrow$ $A = \frac{Px}{2} - x^2 \Rightarrow \frac{dA}{dx} = \frac{P}{2} - 2x$

For maxima or minima, $\frac{dA}{dx} = 0$

$\therefore$ $\frac{P}{2} - 2x = 0 \Rightarrow x = \frac{P}{4}$

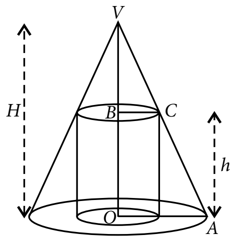
Also, $\frac{d^2A}{dx^2} = -2 < 0$

$\therefore$ Area is maximum, when $x = \frac{P}{4}$.

Now, $y = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$

$\therefore$ Area is maximum, when $x = y$ *i.e.* rectangle is a square.

17. Let R and H be the base radius and height of the given cone and r, h be the same for the inscribed cylinder. Clearly, ΔVBC is similar to ΔVOA.



$\therefore \quad \frac{BV}{BC} = \frac{OV}{OA} \Rightarrow \frac{H-h}{r} = \frac{H}{R}$

$\Rightarrow \quad H - h = H\frac{r}{R} \Rightarrow h = H - H\frac{r}{R}.$

Now, the curved surface area, S of the inscribed cylinder is

$$S = 2\pi rh = 2\pi r\left(H - H\frac{r}{R}\right) = \frac{2\pi H}{R}(rR - r^2)$$

$\Rightarrow \quad \frac{dS}{dr} = \frac{2\pi H}{R}(1\cdot R - 2r)$

For greatest (maximum) or least (minimum) surface area,

$\frac{dS}{dr} = 0 \Rightarrow R = 2r$

Also, $\frac{d^2S}{dr^2} = \frac{2\pi H}{R}\cdot(-2) < 0$

Hence, S is greatest, when $r = \frac{R}{2}$.

18. Let the point on $y^2 = 2x$, which is at a minimum distance from $Q(1, 4)$ be $P(x, y)$.

$\therefore$ We have to minimise

$s = PQ^2 = (x - 1)^2 + (y - 4)^2$

$= \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2 \qquad \left[\because \ y^2 = 2x \Rightarrow x = \frac{1}{2}y^2\right]$

$\Rightarrow \quad \frac{ds}{dy} = 2\left(\frac{y^2}{2} - 1\right)\cdot y + 2(y-4) = y^3 - 8$ and $\frac{d^2s}{dy^2} = 3y^2$

For maxima or minima, $\frac{ds}{dy} = 0$

$\Rightarrow \quad y^3 - 8 = 0 \Rightarrow y^3 - 2^3 = 0$

$\Rightarrow \quad (y - 2)(y^2 + 2y + 4) = 0$

$\Rightarrow \quad y = 2 \qquad (\because y^2 + 2y + 4 = 0$ has no real roots$)$

For this value of y, $\frac{d^2s}{dy^2} = 3 \times 2^2 = 12 > 0$

$\therefore$ s is minimum and from $y^2 = 2x$,

$x = \frac{y^2}{2} = \frac{2^2}{2} = 2$

$\therefore$ The required point is $(2, 2)$.