Integrals

EXAM DRILL

SOLUTIONS

1. (c) : We have, $\int \frac{x+1}{x^{1/2}}dx = \int \sqrt{x}\, dx + \int \frac{1}{\sqrt{x}} dx$

$= \frac{2}{3}x^{3/2} + 2x^{1/2} + c$

2. (a) : Let $I = \int \frac{\sin^2 x}{1+\cos x} dx = \int \frac{1-\cos^2 x}{1+\cos x} dx$

$\Rightarrow \quad = \int(1 - \cos x)dx = x - \sin x + C$

3. (d) : $\int f(x)dx = \int e^{2x}dx = \frac{e^{2x}}{2} + C$

So, option (d) is correct.

4. (a) : We have, $\int (x+2)^7 (x+1)(x+3)dx$

$= \int (x+2)^7 [x^2 + 4x + 4 - 4 + 3] dx$

$= \int (x+2)^7 [(x+2)^2 - 1] dx$

$= \int (x+2)^9 dx - \int (x+2)^7 dx = \frac{(x+2)^{10}}{10} - \frac{(x+2)^8}{8} + C$

5. (b) : Let $I = \int \frac{\sec^8 x}{\operatorname{cosec} x} dx = \int \frac{\sin x}{\cos^8 x} dx$

$= \int \tan x \cdot \sec^7 x\, dx = \int \sec^6 x \cdot \sec x \tan x\, dx$

Put $\sec x = t \Rightarrow \sec x \tan x\, dx = dt$

$\therefore \quad I = \int t^6 dt = \frac{t^7}{7} + c = \frac{\sec^7 x}{7} + c$

6. (d) : Let $f(x) = x(1 - x)(1 + x)$, then

$f(-x) = - x(1 + x)(1- x) = -f(x)$

i.e., $f(x)$ is an odd function.

$\therefore \int_{-1}^{1} x(1-x)(1+x)dx = 0$

(Using property $\int_{-a}^{a} f(x)dx = 0$, if $f(x)$ is an odd function.)

7. (b) : Let $I = \int_{-5}^{5} |x+2|\, dx$

$= -\int_{-5}^{-2} (x+2)dx + \int_{-2}^{5} (x+2)dx$

$= -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5} = \frac{9}{2} + \frac{49}{2} = \frac{58}{2} = 29$

8. (b) : Given, $\int_0^x f(t)\, dt = x^2 + e^x \ (x > 0)$

$\Rightarrow \quad f(x) = 2x + e^x$

$\therefore \quad f(1) = 2(1) + e = 2 + e$

9. (a) : We have, $\int_1^e \log x\, dx$

$= [\log x \cdot x]_1^e - \int_1^e \frac{1}{x} \cdot x\, dx = [e \log e - 1 \log(1)] - \int_1^e 1\, dx$

$= e - [x]_1^e = e - [e - 1] = 1$

10. (d) : $\int_2^4 \log[x]dx = \int_2^3 \log[x]dx + \int_3^4 \log[x]dx$

$= \int_2^3 \log 2\, dx + \int_3^4 \log 3\, dx$

$= (\log 2)(3 - 2) + (\log 3)(4 - 3) = \log 3 + \log 2 = \log 6.$

11. Let $I = \int_0^{\pi/4} e^{\tan x} \sec^2 x\, dx$

Putting $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

When $x = 0$, $t = 0$; when, $x = \frac{\pi}{4}, t = 1$

$\therefore I = \int_0^1 e^t dt = [e^t]_0^1 = e^1 - e^0 = e - 1$

12. Let $I = \int_{1/\pi}^{2\pi} \frac{\sin\left(\frac{1}{x}\right)}{x^2} dx$

Putting $\frac{1}{x} = t \Rightarrow -\frac{1}{x^2}dx = dt$

When $x = \frac{1}{\pi}, \ t = \pi$; when $x = \frac{2}{\pi}, t = \frac{\pi}{2}$

$\therefore I = -\int_{\pi}^{\pi/2} \sin t\, dt = \int_{\pi/2}^{\pi} \sin t\, dt$

$\Rightarrow I = [-\cos t]_{\pi/2}^{\pi} = 1.$

13. Let $I = \int_0^{\pi/2} (\sin x)^{3/2} \cos^3 x\, dx$

Putting $\sin x = t \Rightarrow \cos x\, dx = dt$

When, $x = 0, t = 0$; when $x = \frac{\pi}{2}, \ t = 1$

$\therefore I = \int_0^1 t^{3/2}(1 - t^2)\, dt$

$= \int_0^1 (t^{3/2} - t^{7/2})dt = \left[\frac{2}{5}t^{5/2} - \frac{2}{9}t^{9/2}\right]_0^1$

$= \frac{2}{5} - \frac{2}{9} = \frac{8}{45}$

14. Let $I=\int_0^{\pi/2}\frac{\cos^{1/4}x}{(\sin^{1/4}x+\cos^{1/4}x)}dx$ ...(i)

$$\Rightarrow I=\int_0^{\pi/2}\frac{\cos^{1/4}\left(\frac{\pi}{2}-x\right)}{\sin^{1/4}\left(\frac{\pi}{2}-x\right)+\cos^{1/4}\left(\frac{\pi}{2}-x\right)}dx$$

$$\Rightarrow I=\int_0^{\pi/2}\frac{\sin^{1/4}x}{\sin^{1/4}x+\cos^{1/4}x}dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$\Rightarrow 2I=\int_0^{\pi/2}dx=[x]_0^{\pi/2}$$

$$\Rightarrow 2I=\frac{\pi}{2} \Rightarrow I=\frac{\pi}{4}$$

15. Let $I=\int\frac{\log x}{\log 10}dx=\frac{1}{\log_e 10}\int \log x\,dx$

$$=\log_{10}e\left[\log x\int 1dx-\int\left(\frac{1}{x}\cdot\int 1dx\right)dx\right]$$

$$=\log_{10}e\left[\log x.x-\int\frac{1}{x}.x\,dx\right]$$

$$=\log_{10}e\,[x\log x-x]+c$$

$$=\log_{10}e[\log x-1]x+c$$

16. Let $I=\int_e^{e^2}\frac{dx}{x\log x}$

Putting $\log x = t \Rightarrow \frac{1}{x}dx=dt$

When $x = e$, $t = \log e = 1$;

When $x = e^2$, $t = \log e^2 = 2\log e = 2$

$$\therefore \quad I=\int_1^2\frac{dt}{t}=[\log t]_1^2=\log 2-\log 1=\log 2$$

OR

Let $I=\int\frac{x+\cos 6x}{3x^2+\sin 6x}dx$

Putting $3x^2 + \sin 6x = t \Rightarrow (6x + 6\cos 6x)dx = dt$

$$\therefore I=\frac{1}{6}\int\frac{dt}{t}=\frac{1}{6}\log|t|+C$$

$$\Rightarrow I=\frac{1}{6}\log(3x^2+\sin 6x)+C$$

17. Let $I=\int\frac{2x+3}{x^2+3x}dx$

Putting $x^2 + 3x = t \Rightarrow (2x + 3)dx = dt$

$$\therefore \quad I=\int\frac{1}{t}dt=\log|t|+C=\log|x^2+3x|+C$$

OR

Let $I=\int\tan^2 x\sec^4 x\,dx$

$$\Rightarrow I=\int\tan^2 x(1+\tan^2 x)\sec^2 x\,dx$$

Putting $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\therefore \quad I=\int t^2(1+t^2)dt=\int(t^2+t^4)dt$$

$$\Rightarrow \quad I=\frac{t^3}{3}+\frac{t^5}{5}+C=\frac{\tan^5 x}{5}+\frac{\tan^3 x}{3}+C$$

18. (i) (b) : Given : Let $I=\int xe^{9x^2}dx$

Put $9x^2 = t \Rightarrow 18x\,dx = dt$

$$\therefore \quad I=\int e^t\frac{dt}{18}=\frac{1}{18}e^t+C=\frac{e^{9x^2}}{18}+C$$

(ii) (c) : Given : $\int\frac{1}{x^2+64}dx=a\tan^{-1}\left(\frac{x}{b}\right)+C$

$$=\int\frac{1}{x^2+8^2}dx$$

$$=\frac{1}{8}\tan^{-1}\left(\frac{x}{8}\right)+C \Rightarrow a=\frac{1}{8},\ b=8$$

$$\therefore \quad a+b=\frac{1}{8}+8 \Rightarrow a+b=\frac{65}{8}$$

(iii) (a) : $\int\left(3\sqrt{x}+\frac{4}{x^2}+6\right)dx=64$

$$\Rightarrow \quad \int 3x^{1/2}dx+\int\frac{4}{x^2}dx+\int 6dx=64$$

$$\Rightarrow \quad \frac{3x^{1/2+1}}{\frac{1}{2}+1}+\frac{4x^{-2+1}}{-2+1}+6x+C=64$$

$$\Rightarrow \quad 2x^{\frac{3}{2}}-\frac{4}{x}+6x+C=64$$

At $x = 4$, we get

$$2(4)^{3/2}-\frac{4}{4}+6(4)+C=64 \Rightarrow C=25$$

(iv) (d) : By the first fundamental theorem of Integral calculus

$$F'(t)=\frac{d}{dt}\int_0^t(4x^6+1)dt=4t^6+1$$

(v) (a) : $\frac{2x-3}{(x+2)(x-3)}=\frac{A}{(x+2)}+\frac{B}{(x-3)}$

$(2x - 3) = A(x - 3) + B(x + 2)$ ...(i)

Putting $x = 3$ in (i), we get

$2(3) - 3 = B(3 + 2)$

$6 - 3 = 5B$

$$\Rightarrow \quad B=\frac{3}{5}$$

Putting $x = -2$ in (i), we get

$2(-2) -3 = A(-2 -3)$

$-7 = -5A$

$$\Rightarrow \quad A=\frac{7}{5}$$

$$\therefore \quad A+B=\frac{7}{5}+\frac{3}{5}=2$$

19. (i) Solving Ananya's problem by partial fraction, we get

$$\frac{x}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$$

$x = A(x + 2) + B(x + 1)$

Putting $x = -2$, we get

$-2 = B(-2 + 1) \Rightarrow B = 2$

Putting $x = -1$, we get

$-1 = A(-1 + 2) \Rightarrow A = -1$

Now, solve the Sanya's Problem :

$$\frac{y+1}{(y+1)^2(y+3)} = \frac{C}{y+1} + \frac{D}{(y+1)^2} + \frac{E}{(y+3)}$$

$y + 1 = C(y + 1)(y + 3) + D(y + 3) + E(y + 1)^2$

Putting $y = -3$, we get

$$-3 + 1 = E\,(-3 + 1)^2 \Rightarrow E = \frac{-1}{2}$$

Putting $y = -1$, we get

$-1 + 1 = D\,(-1 + 3) \Rightarrow D = 0$

Putting y = 0, we get

$1 = 3C + 3D + 5$

$$\Rightarrow \quad C = \frac{1}{2}$$

$$\therefore \quad A + B + C + D + E = -1 + 2 + \frac{1}{2} + 0 - \frac{1}{2}$$

(ii) Let $I = \int \frac{y+1}{(y+1)^2(y+3)}\,dy$

Let $\frac{y+1}{(y+1)^2(y+3)} = \frac{A}{y+1} + \frac{B}{(y+1)^2} + \frac{C}{(y+3)}$

$y + 1 = A(y + 1)(y + 3) + B(y + 3) + C(y + 1)^2$

Putting $y = -3$, we get

$$C = \frac{-1}{2}$$

Putting $y = -1$, we get

$B = 0$

Putting $y = 0$, we get

$3A + 3B + C = 1$

$$\Rightarrow \quad A = \frac{1}{2}$$

$$\therefore \quad I = \int \frac{1}{2(y+1)}\,dx + \int \frac{0}{(y+1)^2}\,dy - \int \frac{1}{2(y+3)}$$

$$= \frac{1}{2}\log|y+1| - \frac{1}{2}|y+3| + C$$

$$= \frac{\log|y+1| - \log|y+3|}{2} + C$$

20. Let $I = \int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}}\,dx$

Putting $\sin x = t \Rightarrow \cos x\,dx = dt$, we get

$$I = \int \frac{dt}{\sqrt{t^2 - 2t - 3}} = \int \frac{dt}{\sqrt{(t-1)^2 - 2^2}}$$

$$= \log\left|(t-1) + \sqrt{(t-1)^2 - 2^2}\right| + C$$

$$= \log\left|(\sin x - 1) + \sqrt{\sin^2 x - 2\sin x - 3}\right| + C$$

OR

Let $I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}\,dx$

Putting $x^{3/2} = t \Rightarrow \frac{3}{2}x^{1/2}dx = dt$

$$\therefore \quad I = \frac{2}{3}\int \frac{dt}{\sqrt{a^3 - t^2}} = \frac{2}{3}\int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}}$$

$$= \frac{2}{3}\left[\sin^{-1}\left(\frac{t}{a^{3/2}}\right)\right] + C = \frac{2}{3}\left[\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)\right] + C$$

$$= \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + C$$

21. Let $I = \int \sqrt{\frac{a+x}{a-x}}\,dx$

Putting $x = a\cos 2\theta$

$$\Rightarrow \quad \cos 2\theta = \frac{x}{a} \Rightarrow 2\theta = \cos^{-1}\frac{x}{a} \Rightarrow \theta = \frac{1}{2}\cos^{-1}\frac{x}{a}$$

Also, $dx = -2a \cdot \sin 2\theta \cdot d\theta$

$$\therefore \quad I = -2\int \sqrt{\frac{a + a\cos 2\theta}{a - a\cos 2\theta}} \cdot a\sin 2\theta\,d\theta$$

$$= -2a\int \sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\sin 2\theta\,d\theta$$

$$= -2a\int \sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}\sin 2\theta\,d\theta$$

$[\because \cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A]$

$$= -2a\int \frac{\cos\theta}{\sin\theta} \cdot 2\sin\theta\cos\theta\,d\theta$$

$$= -4a\int \cos^2\theta\,d\theta = -2a\int (1 + \cos 2\theta)\,d\theta$$

$$= -2a\left[\int 1\,d\theta + \int \cos 2\theta\,d\theta\right]$$

$$= -2a\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= -2a\left[\frac{1}{2}\cos^{-1}\frac{x}{a} + \frac{1}{2}\sqrt{1 - \frac{x^2}{a^2}}\right] + C$$

$$= -a\left[\cos^{-1}\left(\frac{x}{a}\right) + \sqrt{1 - \frac{x^2}{a^2}}\right] + C$$

22. Let $I = \int \frac{\sqrt{1+x^2}}{x^4}\,dx = \int \frac{\sqrt{1+x^2}}{x} \cdot \frac{1}{x^3}\,dx$

$$= \int \sqrt{\frac{1+x^2}{x^2}} \cdot \frac{1}{x^3}\,dx = \int \sqrt{\frac{1}{x^2} + 1} \cdot \frac{1}{x^3}\,dx$$

Putting $1 + \frac{1}{x^2} = t^2 \Rightarrow -\frac{1}{x^3}\,dx = t\,dt$

Hence, $I = -\int t^2 dt = -\frac{t^3}{3} + C$

$$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2} + C$$

OR

Let $I = \int \frac{dt}{\sqrt{3t-2t^2}} = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left(t^2-\frac{3}{2}t\right)}}$

$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left[\left(t^2-\frac{3}{2}t\right)+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2-\left(t-\frac{3}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{t-\frac{3}{4}}{\frac{3}{4}}\right)+C = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4t-3}{3}\right)+C$$

23. Let $I = \int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$

$$= \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2-(x^{3/2})^2}}dx$$

Putting $x^{3/2} = t \Rightarrow \frac{3}{2}x^{1/2}dx = dt$

$\therefore \quad I = \frac{2}{3}\int \frac{dt}{\sqrt{(a^{3/2})^2-t^2}} = \frac{2}{3}\sin^{-1}\frac{t}{a^{3/2}}+C$

$$= \frac{2}{3}\sin^{-1}\frac{x^{3/2}}{a^{3/2}}+C = \frac{2}{3}\sin^{-1}\sqrt{\frac{x^3}{a^3}}+C$$

24. Let $I = \int_0^{\pi}\frac{x}{1+\sin x}dx$ ... (i)

Using property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$I = \int_0^{\pi}\frac{\pi-x}{1+\sin(\pi-x)}dx = \int_0^{\pi}\frac{\pi-x}{1+\sin x}dx$ ...(ii)

On adding (i) and (ii), we get

$2I = \pi\int_0^{\pi}\frac{1}{1+\sin x}dx$

$= \pi\int_0^{\pi}\frac{(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$

$= \pi\int_0^{\pi}\frac{(1-\sin x)}{\cos^2 x}dx$ [$\because \cos^2 x = 1-\sin^2 x$]

$= \pi\int_0^{\pi}(\sec^2 x - \tan x\cdot\sec x)dx$

$= \pi\left[\tan x - \sec x\right]_0^{\pi}$

$= \pi[\tan\pi - \sec\pi - \tan 0 + \sec 0]$

$\Rightarrow \quad 2I = \pi[0+1-0+1] = 2\pi$

$\therefore \quad I = \pi$

25. Let $I = \int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}d\theta$

$= 3\int\frac{\sin\theta\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta - 2\int\frac{\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta$

$= 3I_1 - 2I_2$ (say) ...(i)

Now, $I_1 = \int\frac{\sin\theta\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta$

Putting $\sin^2\theta = t \Rightarrow 2\sin\theta\cos\theta\, d\theta = dt$

$\therefore \quad I_1 = \frac{1}{2}\int\frac{dt}{4+t-4\sqrt{t}} = \frac{1}{2}\int\frac{dt}{(\sqrt{t}-2)^2}$

Putting $\sqrt{t}-2 = u \Rightarrow \sqrt{t} = u+2$

$\Rightarrow \quad \frac{1}{2\sqrt{t}}dt = du \Rightarrow dt = 2(u+2)du$

$\therefore \quad I_1 = \int\frac{(u+2)}{u^2}du = \int\frac{du}{u}+2\int\frac{du}{u^2}$

$= \log u - \frac{2}{u}+C_1 = \log(\sqrt{t}-2)-\frac{2}{\sqrt{t}-2}+C_1$

$\Rightarrow \quad I_1 = \log(\sin\theta-2)-\frac{2}{\sin\theta-2}+C_1$ ...(ii)

Also, $I_2 = \int\frac{\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta$

Putting $\sin\theta = m \Rightarrow \cos\theta\, d\theta = dm$

$\therefore \quad I_2 = \int\frac{dm}{4+m^2-4m} = \int\frac{dm}{(m-2)^2}$

$= \frac{-1}{m-2}+C_2 = \frac{-1}{\sin\theta-2}+C_2$ ...(iii)

From (i), (ii) and (iii), we get

$I = 3\log(\sin\theta-2)-\frac{6}{\sin\theta-2}+\frac{2}{\sin\theta-2}+C,$

where $C = 3C_1 - 2C_2$

$\Rightarrow \quad I = 3\log(\sin\theta-2)-\frac{4}{\sin\theta-2}+C\,]$

OR

Let $I = \int\frac{\sin^6 x+\cos^6 x}{\sin^2 x\cdot\cos^2 x}dx$

$= \int\frac{\sin^6 x}{\sin^2 x\cdot\cos^2 x}dx + \int\frac{\cos^6 x}{\sin^2 x\cdot\cos^2 x}dx$

$$= \int \frac{\sin^4 x}{\cos^2 x} dx + \int \frac{\cos^4 x}{\sin^2 x} dx$$

$$= \int \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x} dx + \int \frac{\cos^2 x(1-\sin^2 x)}{\sin^2 x} dx$$

$= \int \tan^2 x\, dx - \int \sin^2 x\, dx + \int \cot^2 x\, dx - \int \cos^2 x\, dx$

$= \int (\sec^2 x - 1) dx + \int (\text{cosec}^2 x - 1)\, dx - \int \sin^2 x\, dx - \int (1 - \sin^2 x)\, dx$

$= \tan x - x + (-\cot x) - x - x + C = \tan x - \cot x - 3x + C$

26. Let $I = \int \frac{x}{(x^2+1)(x-1)} dx$

Let $\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$ ...(1)

$\Rightarrow \quad x = (Ax + B)(x - 1) + C(x^2 + 1)$ ...(2)

Comparing coefficients of x^2, x and constant terms, we get

$0 = A + C;\ 1 = B - A;\ 0 = -B + C$

Solving these, we get

$A = -\frac{1}{2}, C = \frac{1}{2}, B = \frac{1}{2}$

$\therefore$ From (1), we get

$$\frac{x}{(x^2+1)(x-1)} = \frac{-\frac{1}{2}(x-1)}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x-1}$$

$$= -\frac{1}{2}\cdot\frac{x}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x-1}$$

$$\therefore \quad I = -\frac{1}{4}\int \frac{2x}{x^2+1} dx + \frac{1}{2}\int \frac{dx}{x^2+1} + \frac{1}{2}\int \frac{dx}{x-1}$$

$$\Rightarrow \quad I = -\frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1} x + \frac{1}{2}\log|x-1| + C_1$$

OR

Let $I = \int \frac{x^3}{x^4+3x^2+2} dx$

Put $x^2 = t \Rightarrow x\,dx = \frac{1}{2} dt$

$$\therefore \quad I = \frac{1}{2}\int \frac{t}{t^2+3t+2} dt = \frac{1}{2}\int \frac{t}{(t+2)(t+1)} dt$$

Let $\frac{t}{(t+2)(t+1)} = \frac{A}{(t+2)} + \frac{B}{(t+1)}$

$\Rightarrow \quad t = A(t + 1) + B(t + 2)$ ...(i)

Put $t = -1, -2$ in (i), we get $A = 2, B = -1$

$$\therefore \quad \frac{t}{(t+2)(t+1)} = \frac{2}{t+2} - \frac{1}{t+1}$$

$$\Rightarrow \quad I = \frac{1}{2}\int \left[\frac{2}{(t+2)} - \frac{1}{(t+1)}\right] dt$$

$$= \frac{1}{2}[2\log|t+2| - \log|t+1|] + C$$

$$= \frac{1}{2}[2\log|x^2+2| - \log|x^2+1|] + C$$

27. Let $I = \int_0^{\pi/2} \frac{\tan x}{1+m^2\tan^2 x} dx$

$$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1+m^2\cdot\frac{\sin^2 x}{\cos^2 x}} dx$$

$$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{\frac{\cos^2 x + m^2\sin^2 x}{\cos^2 x}} dx$$

$$= \int_0^{\pi/2} \frac{\sin x\cos x}{1-\sin^2 x(1-m^2)} dx \qquad [\because \cos^2 x = 1 - \sin^2 x]$$

Putting $\sin^2 x = t \Rightarrow 2\sin x \cos x\, dx = dt$

When $x = 0$, $t = 0$; when $x = \frac{\pi}{2}, t = 1$

$$\therefore \quad I = \frac{1}{2}\int_0^1 \frac{dt}{1-t(1-m^2)}$$

$$\Rightarrow I = \frac{1}{2}\left[-\log|1-t(1-m^2)|\cdot\frac{1}{1-m^2}\right]_0^1$$

$$\Rightarrow I = \frac{1}{2}\left[-\log|1-1+m^2|\cdot\frac{1}{1-m^2} + \log|1|\cdot\frac{1}{1-m^2}\right]$$

$$\Rightarrow I = \frac{1}{2}\left[-\log|m^2|\cdot\frac{1}{1-m^2}\right] \qquad [\because \log 1 = 0]$$

$$\Rightarrow \quad I = \frac{2}{2}\cdot\frac{\log m}{(m^2-1)} = \frac{\log m}{m^2-1}$$

28. Let $I = \int \frac{(2x-1)}{(x-1)(x+2)(x-3)} dx$

By using partial fraction, we get

$$\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x-3)}$$

$\Rightarrow \quad 2x - 1 = A(x + 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x + 2)$ ...(i)

Putting $x = 1$ in (i) we get

$2 - 1 = A(1 + 2)(1 - 3)$

$\Rightarrow \quad 1 = -6A \quad \Rightarrow \quad A = -\frac{1}{6}$

Putting $x = 3$ in (i), we get

$6 - 1 = C(3 - 1)(3 + 2)$

$\Rightarrow \quad 5 = 10C \quad \Rightarrow \quad C = \frac{1}{2}$

Now, putting $x = -2$ in (i), we get

$-4 - 1 = B(-2 - 1)(-2 - 3)$

$$\Rightarrow \quad -5 = 15B \quad \Rightarrow \quad B = -\frac{1}{3}$$

$$\therefore \quad I = -\frac{1}{6}\int\frac{1}{x-1}dx - \frac{1}{3}\int\frac{1}{x+2}dx + \frac{1}{2}\int\frac{1}{x-3}dx$$

$$= -\frac{1}{6}\log|(x-1)| - \frac{1}{3}\log|(x+2)| + \frac{1}{2}\log|(x-3)| + C$$

$$= -\log|(x-1)|^{1/6} - \log|(x+2)|^{1/3} + \log\left|\sqrt{(x-3)}\right| + C$$

$$= \log\left|\sqrt{x-3}\right| - \log\left|(x-1)^{1/6}(x+2)^{1/3}\right| + C$$

$$= \log\left|\frac{\sqrt{x-3}}{(x-1)^{1/6}(x+2)^{1/3}}\right| + C$$

OR

Let $I = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}dx$

$$= \int_{\pi/3}^{\pi/2} \frac{\left(2\cos^2\frac{x}{2}\right)^{1/2}}{\left(2\sin^2\frac{x}{2}\right)^{5/2}}dx$$

[Using $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$]

$$= \frac{\sqrt{2}}{4\sqrt{2}}\int_{\pi/3}^{\pi/2}\frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}dx = \frac{1}{4}\int_{\pi/3}^{\pi/2}\frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}dx$$

Putting $\sin\frac{x}{2} = t \quad \Rightarrow \quad \cos\frac{x}{2}dx = 2dt$

When, $x = \frac{\pi}{3} \Rightarrow t = \frac{1}{2}$; when $x = \frac{\pi}{2} \Rightarrow t = \frac{1}{\sqrt{2}}$

$$\therefore \quad I = \frac{2}{4}\int_{1/2}^{1/\sqrt{2}}\frac{dt}{t^5} = \frac{1}{2}\left[\frac{t^{-4}}{-4}\right]_{1/2}^{1/\sqrt{2}}$$

$$= -\frac{1}{8}\left[\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^4} - \frac{1}{\left(\frac{1}{2}\right)^4}\right] = -\frac{1}{8}(4-16) = \frac{12}{8} = \frac{3}{2}$$

29. Let $I = \int_0^{\pi} x\log\sin x\,dx$ ... (i)

$$\Rightarrow \quad I = \int_0^{\pi}(\pi - x)\log\sin(\pi - x)dx$$

$\left[\because \int_0^a f(x)dx = \int_0^a (a-x)dx\right]$

$$\Rightarrow \quad I = \int_0^{\pi}(\pi - x)\log\sin x\,dx \quad \text{... (ii)}$$

[$\because \sin(\pi - \theta) = \sin\theta$]

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi}\log\sin x\,dx \quad \text{... (*)}$$

$$\Rightarrow \quad 2I = 2\pi\int_0^{\pi/2}\log\sin x\,dx$$

$\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x)\right]$

$$\Rightarrow \quad I = \pi\int_0^{\pi/2}\log\sin x\,dx \quad \text{... (iii)}$$

$$\Rightarrow \quad I = \pi\int_0^{\pi/2}\log\sin(\pi/2 - x)dx$$

$$\Rightarrow I = \pi\int_0^{\pi/2}\log\cos x\,dx \quad \text{... (iv)}$$

On adding (iii) and (iv), we get

$$2I = \pi\int_0^{\pi/2}(\log\sin x + \log\cos x)dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi/2}\log\sin x\cos x\,dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi/2}\log\frac{2\sin x\cos x}{2}dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi/2}(\log\sin 2x - \log 2)\,dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi/2}\log\sin 2x\,dx - \pi\int_0^{\pi/2}\log 2\,dx$$

Putting $2x = t \quad \Rightarrow \quad dx = \frac{1}{2}dt$

When, $x = 0 \Rightarrow t = 0$; when $x = \frac{\pi}{2} \Rightarrow t = \pi$

$$\therefore \quad 2I = \frac{\pi}{2}\int_0^{\pi}\log\sin t\,dt - \frac{\pi^2}{2}\log 2$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\pi}\log\sin x\,dx - \frac{\pi^2}{2}\log 2$$

$$\Rightarrow \quad 2I = I - \frac{\pi^2}{2}\log 2 \quad \text{[from (*)]}$$

$$\therefore \quad I = -\frac{\pi^2}{2}\log 2 = \frac{\pi^2}{2}\log\left(\frac{1}{2}\right)$$

30. Let $I = \int_0^1 x\log(1+2x)\,dx$

$$= \left[\log(1+2x)\frac{x^2}{2}\right]_0^1 - \int_0^1\frac{1}{1+2x}\cdot 2\cdot\frac{x^2}{2}dx$$

[Integration by parts]

$$= \frac{1}{2}[x^2\log(1+2x)]_0^1 - \int_0^1\frac{x^2}{1+2x}dx$$

$$=\frac{1}{2}[1\log 3-0]-\left[\int_0^1\left(\frac{x}{2}-\frac{\frac{x}{2}}{1+2x}\right)dx\right]$$

$$\Rightarrow I=\frac{1}{2}\log 3-\frac{1}{2}\int_0^1 x\,dx+\frac{1}{2}\int_0^1\frac{x}{1+2x}dx$$

$$=\frac{1}{2}\log 3-\frac{1}{2}\left[\frac{x^2}{2}\right]_0^1+\frac{1}{2}\int_0^1\frac{\frac{1}{2}(2x+1-1)}{(2x+1)}dx$$

$$=\frac{1}{2}\log 3-\frac{1}{2}\left[\frac{1}{2}-0\right]+\frac{1}{4}\int_0^1 dx-\frac{1}{4}\int_0^1\frac{1}{1+2x}dx$$

$$=\frac{1}{2}\log 3-\frac{1}{4}+\frac{1}{4}[x]_0^1-\frac{1}{8}[\log|(1+2x)|]_0^1$$

$$=\frac{1}{2}\log 3-\frac{1}{4}+\frac{1}{4}-\frac{1}{8}[\log 3-\log 1]$$

$$=\frac{1}{2}\log 3-\frac{1}{8}\log 3=\frac{3}{8}\log 3$$

OR

Let $I=\int\frac{x^3}{x^{16}+4}dx$

Putting $x^4=t\Rightarrow 4x^3\,dx=dt$, we get

$$I=\frac{1}{4}\int\frac{1}{t^4+4}dt=\frac{1}{4}\int\frac{\frac{1}{t^2}}{t^2+\frac{4}{t^2}}dt$$

$$\Rightarrow I=\frac{1}{16}\int\frac{\frac{4}{t^2}}{t^2+\frac{4}{t^2}}dt=\frac{1}{16}\int\frac{\left(1+\frac{2}{t^2}\right)-\left(1-\frac{2}{t^2}\right)}{t^2+\frac{4}{t^2}}dt$$

$$\Rightarrow I=\frac{1}{16}\int\frac{1+\frac{2}{t^2}}{t^2+\frac{4}{t^2}}dt-\frac{1}{16}\int\frac{1-\frac{2}{t^2}}{t^2+\frac{4}{t^2}}dt$$

$$\Rightarrow I=\frac{1}{16}\int\frac{1+\frac{2}{t^2}}{\left(t-\frac{2}{t}\right)^2+4}dt-\frac{1}{16}\int\frac{1-\frac{2}{t^2}}{\left(t+\frac{2}{t}\right)^2-4}dt$$

$$\Rightarrow I=\frac{1}{16}\int\frac{du}{u^2+2^2}-\frac{1}{16}\int\frac{dv}{v^2-2^2},$$

where $u=t-\frac{2}{t}$ and $v=t+\frac{2}{t}$

$$\Rightarrow I=\frac{1}{32}\tan^{-1}\left(\frac{u}{2}\right)-\frac{1}{16}\times\frac{1}{2\times 2}\log\left|\frac{v-2}{v+2}\right|+C$$

$$\Rightarrow I=\frac{1}{32}\tan^{-1}\left(\frac{t^2-2}{2t}\right)-\frac{1}{64}\log\left|\frac{t^2-2t+2}{t^2+2t+2}\right|+C$$

$$\Rightarrow I=\frac{1}{32}\tan^{-1}\left(\frac{x^8-2}{2x^4}\right)-\frac{1}{64}\log\left|\frac{x^8-2x^4+2}{x^8+2x^4+2}\right|+C$$

31. Let $I=\int\frac{\sin x}{\sin 4x}dx$

$$=\int\frac{\sin x}{2\sin 2x\cos 2x}dx$$

$$=\int\frac{\sin x}{4\sin x\cos x\cos 2x}dx$$

$$\Rightarrow I=\frac{1}{4}\int\frac{1}{\cos x\cos 2x}dx=\frac{1}{4}\int\frac{\cos x}{\cos^2 x\cos 2x}dx$$

$$\Rightarrow I=\frac{1}{4}\int\frac{\cos x}{(1-\sin^2 x)(1-2\sin^2 x)}dx$$

Putting $\sin x=t\Rightarrow\cos x\,dx=dt$, we get

$$I=\frac{1}{4}\int\frac{dt}{(1-t^2)(1-2t^2)}$$

Let $t^2=y$.

Then, $\frac{1}{(1-t^2)(1-2t^2)}=\frac{1}{(1-y)(1-2y)}$

Let $\frac{1}{(1-y)(1-2y)}=\frac{A}{1-y}+\frac{B}{1-2y}$.

Then, $1=A(1-2y)+B(1-y)$ ...(i)

Putting $y=1$ and $y=\frac{1}{2}$ respectively in (i), we get

$A=-1$ and $B=2$.

$$\therefore\ \frac{1}{(1-y)(1-2y)}=\frac{-1}{1-y}+\frac{2}{1-2y}$$

$$\Rightarrow\ \frac{1}{(1-t^2)(1-2t^2)}=-\frac{1}{1-t^2}+\frac{2}{1-2t^2}$$

$$\Rightarrow\ I=\frac{1}{4}\int\left(-\frac{1}{1-t^2}+\frac{2}{1-2t^2}\right)dt$$

$$\Rightarrow\ I=-\frac{1}{4}\int\frac{1}{1-t^2}dt+\frac{2}{4}\int\frac{1}{1-(\sqrt{2}t)^2}dt$$

$$\Rightarrow\ I=-\frac{1}{4}\times\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|+\frac{1}{2}\times\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}t}{1-\sqrt{2}t}\right|+C$$

$$\Rightarrow\ I=-\frac{1}{8}\log\left|\frac{1+t}{1-t}\right|+\frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}t}{1-\sqrt{2}t}\right|+C$$

$$\Rightarrow\ I=-\frac{1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right|+\frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|+C$$

32. Let $I=\int_{-2}^{2}|x\cos\pi x|\,dx$ and $f(x)=|x\cos\pi x|$. Then,

$f(-x)=|-x\cos(-\pi x)|=|-x\cos\pi x|=|x\cos\pi x|=f(x)$

So, $f(x)$ is an even function.

$\therefore\ I = 2\int_0^2 |x\cos \pi x|\,dx$

$$\left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(-x) = f(x)\right]$$

Now, $f(x) = |x\cos\pi x| = \begin{cases} x\cos\pi x, \text{ if } 0 \le x \le \frac{1}{2} \\ -x\cos\pi x, \text{ if } \frac{1}{2} < x < \frac{3}{2} \\ x\cos\pi x, \text{ if } \frac{3}{2} \le x \le 2 \end{cases}$

$\therefore\ I = 2\int_0^2 |x\cos\pi x|\,dx$

$\Rightarrow I = 2\left\{\int_0^{1/2} x\cos\pi x\,dx + \int_{1/2}^{3/2} -x\cos\pi x\,dx + \int_{3/2}^{2} x\cos\pi x\,dx\right\}$

[Using additive property]

$$\Rightarrow I = 2\left\{\left[\frac{x}{\pi}\sin\pi x + \frac{1}{\pi^2}\cos\pi x\right]_0^{1/2} - \left[\frac{x}{\pi}\sin\pi x + \frac{1}{\pi^2}\cos\pi x\right]_{1/2}^{3/2} + \left[\frac{x}{\pi}\sin\pi x + \frac{1}{\pi^2}\cos\pi x\right]_{3/2}^{2}\right\}$$

$$\Rightarrow I = 2\left[\left\{\left(\frac{1}{2\pi}+0\right) - 0\left(0+\frac{1}{\pi^2}\right)\right\} - \left\{\left(-\frac{3}{2\pi}+0\right) - \left(\frac{1}{2\pi}+0\right)\right\} + \left\{\left(0+\frac{1}{\pi^2}\right) - \left(-\frac{3}{2\pi}+0\right)\right\}\right]$$

$$\Rightarrow I = 2\left\{\left(\frac{1}{2\pi} - \frac{1}{\pi^2}\right) + \left(\frac{3}{2\pi} + \frac{1}{2\pi}\right) + \left(\frac{1}{\pi^2} + \frac{3}{2\pi}\right)\right\} = \frac{8}{\pi}$$

33. Let $I = \int_{-\pi/4}^{\pi/4} \frac{x + \frac{\pi}{4}}{2 - \cos 2x}\,dx.$

$$\Rightarrow I = \int_{-\pi/4}^{\pi/4} \frac{x}{2-\cos 2x}\,dx + \frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{1}{2-\cos 2x}\,dx \quad \text{...(i)}$$

We observe that $\frac{x}{2-\cos 2x}$ is an odd function and $\frac{1}{2-\cos 2x}$ is an even function.

$\therefore\ \int_{-\pi/4}^{\pi/4} \frac{x}{2-\cos 2x}\,dx = 0$

and $\int_{-\pi/4}^{\pi/4} \frac{1}{2-\cos 2x}\,dx = 2\int_0^{\pi/4} \frac{1}{2-\cos 2x}\,dx.$

Substituting these values in (i), we get

$I = 0 + 2\left(\frac{\pi}{4}\right)\int_0^{\pi/4} \frac{1}{2-\cos 2x}\,dx$

$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi/4} \frac{1+\tan^2 x}{1+3\tan^2 x}\,dx$

$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2 x}{1^2 + (\sqrt{3}\tan x)^2}\,dx$

Put $\sqrt{3}\tan x = t \Rightarrow \sqrt{3}\sec^2 x\,dx = dt$

When, $x = 0,\ t = \sqrt{3}\tan 0 = 0$;

when, $x = \frac{\pi}{4},\ t = \sqrt{3}\tan\frac{\pi}{4} = \sqrt{3}.$

$\therefore\ I = \frac{\pi}{2\sqrt{3}}\int_0^{\sqrt{3}} \frac{1}{1^2+t^2}\,dt = \frac{\pi}{2\sqrt{3}}[\tan^{-1} t]_0^{\sqrt{3}}$

$= \frac{\pi}{2\sqrt{3}}(\tan^{-1}\sqrt{3} - \tan^{-1} 0)$

$\Rightarrow I = \frac{\pi}{2\sqrt{3}} \times \frac{\pi}{3} = \frac{\pi^2}{6\sqrt{3}}$

OR

Let $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx.$

$\Rightarrow\ I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$

$\Rightarrow\ I = I_1 + I_2$, where

$I_1 = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx$ and $I_2 = \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$

Let $f(x) = \frac{2x}{1+\cos^2 x}$ is an odd function and

Let $g(x) = \frac{2x\sin x}{1+\cos^2 x}$ is an even function.

$\therefore\ I_1 = \int_{-\pi}^{\pi} f(x)\,dx = 0$ and,

$I_2 = \int_{-\pi}^{\pi} g(x)dx = 2\int_0^{\pi} g(x)\,dx = 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx.$

Now, $I_2 = 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

$\Rightarrow I_2 = 4\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$

$$\left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$\Rightarrow\ I_2 = 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

$\Rightarrow I_2 = 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx - I_2$

$\Rightarrow\ 2I_2 = 4\pi\int_0^{\pi} \frac{1}{1+\cos^2 x}\sin x\,dx$

$\Rightarrow\ 2I_2 = -4x\pi\int_1^{-1} \frac{1}{1+t^2}\,dt$ (where $t = \cos x$)

$\Rightarrow\ 2I_2 = -4\pi[\tan^{-1} t]_1^{-1} = -4\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = 2\pi^2$

$\Rightarrow\ I_2 = \pi^2$