CHAPTER 7

Integrals

TRY YOURSELF

SOLUTIONS

1. (i) $\int x^{99} dx = \frac{x^{99+1}}{99+1} + C = \frac{x^{100}}{100} + C$

(ii) $\int \frac{1}{\sqrt[3]{x}} dx = \int \frac{1}{x^{1/3}} dx = \int x^{-1/3} dx$

$= \frac{x^{-(1/3)+1}}{\frac{-1}{3}+1} + C = \frac{3}{2} x^{2/3} + C$

(iii) $\int \frac{\sin x}{1-\sin^2 x} dx = \int \frac{\sin x}{\cos^2 x} dx$

$= \int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} dx = \int \sec x \tan x \, dx = \sec x + C$

2. (i) $\int 3^x e^x dx = \int (3e)^x dx = \frac{(3e)^x}{\log 3e} + C$

(ii) $\int \sin^2 x \operatorname{cosec}^2 x \, dx = \int \frac{\sin^2 x}{\sin^2 x} dx = \int 1 dx = x + c$

3. (i) $\int \left(x - \frac{1}{x}\right)^3 dx = \int \left(x^3 - \frac{1}{x^3} - 3x \cdot \frac{1}{x}\left(x - \frac{1}{x}\right)\right) dx$

$= \int \left(x^3 - x^{-3} - 3x + \frac{3}{x}\right) dx$

$= \frac{x^4}{4} - \frac{x^{-2}}{-2} - \frac{3x^2}{2} + 3\log|x| + C$

$= \frac{x^4}{4} + \frac{1}{2x^2} - \frac{3}{2}x^2 + 3\log|x| + C$

(ii) $\int \frac{1+\sin x}{1-\sin x} dx = \int \frac{1+\sin x}{1-\sin x} \times \frac{1+\sin x}{1+\sin x} dx$

$= \int \frac{(1+\sin x)^2}{1-\sin^2 x} dx$

$= \int \frac{1+\sin^2 x + 2\sin x}{\cos^2 x} dx$

$= \int \sec^2 x + \tan^2 x + 2\sec x \tan x dx$

$= \int (\sec^2 x + (\sec^2 x - 1) + 2\sec x \tan x) dx$

$= \int (2\sec^2 x + 2\sec x \tan x - 1) dx$

$= 2\tan x + 2\sec x - x + c$

(iii)

$\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^3 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^3 x}{\sin^2 x \cos^2 x} dx$

$= \int \frac{\sin x}{\cos^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx$

$= \int \tan x \sec x \, dx + \int \cot x \operatorname{cosec} x \, dx$

$= \sec x - \operatorname{cosec} x + c$

4. We have, $f(x) = \int f'(x) dx = \int \left(3x^2 - \frac{2}{x^3}\right) dx$

$\Rightarrow f(x) = \frac{3x^3}{3} - 2\frac{x^{-2}}{-2} + C = x^3 + \frac{1}{x^2} + C$

$f(1) = 1 + 1 + C \Rightarrow 0 = 2 + C \Rightarrow C = -2$

$\therefore f(x) = x^3 + \frac{1}{x^2} - 2.$

5. (i) Let $I = \int \cos(ax+b) dx$

Put $(ax+b) = t \Rightarrow a dx = dt \Rightarrow dx = \frac{1}{a} dt$

$\therefore I = \frac{1}{a}\int \cos t \, dt = \frac{1}{a}\sin t + C = \frac{1}{a}\sin(ax+b) + C$

(ii) Let $I = \int \frac{1-\tan x}{1+\tan x} dx$

$= \int \frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}} dx = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

Put $(\cos x + \sin x) = t$

$\Rightarrow$ $(-\sin x + \cos x) dx = dt \Rightarrow (\cos x - \sin x) dx = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + C = \log|\cos x + \sin x| + C$

6. (i) Let $I = \int x^3 \sin^4(x^4)\cos(x^4) dx$

Put $\sin(x^4) = t$

$\Rightarrow 4x^3 \cos(x^4) dx = dt$

$\Rightarrow x^3 \cos(x^4) dx = \frac{dt}{4}$

$\therefore I = \frac{1}{4}\int t^4 dt = \frac{1}{4}\frac{t^5}{5} + C$

$= \frac{1}{20}\sin^5(x^4) + C$

(ii) Let $I = \int (5x+3)\sqrt{2x-1} \, dx$

Here $\sqrt{2x-1}$ is of the form $(ax+b)^n$

Put $2x - 1 = t \Rightarrow 2dx = dt \Rightarrow \frac{dt}{2} = dx$

Also, $2x - 1 = t \Rightarrow x = \frac{t+1}{2}$

$\therefore I = \int \left(5\left(\frac{t+1}{2}\right) + 3\right)\sqrt{t} \frac{dt}{2}$

$$= \int \frac{5t+11}{4}\sqrt{t}\,dt = \int\left(\frac{5}{4}t^{3/2} + \frac{11}{4}t^{1/2}\right)dt$$

$$= \frac{5}{4}\frac{t^{5/2}}{\frac{5}{2}} + \frac{11}{4}\frac{t^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{1}{2}t^{5/2} + \frac{11}{6}t^{3/2} + C$$

$$= \frac{t^{3/2}}{2}\left(t + \frac{11}{3}\right) + C = \frac{(2x-1)^{3/2}}{2}\left(\frac{3(2x-1)+11}{3}\right) + C$$

$$= \frac{(2x-1)^{3/2}}{6}[3(2x-1)+11] + C$$

$$= \frac{(2x-1)^{3/2}}{6}[6x+8] + C = \frac{(2x-1)^{3/2}(3x+4)}{3} + C$$

(iii) Let $I = \int \frac{e^{2x}}{1+e^x}dx = \int \frac{e^x}{1+e^x}e^x dx$

Put $1 + e^x = t \Rightarrow e^x dx = dt$

Also, $1 + e^x = t \Rightarrow e^x = t - 1$

$$\therefore\ I = \int \frac{t-1}{t}dt = \int\left(1 - \frac{1}{t}\right)dt$$

$= t - \log|t| + C$

$= 1 + e^x - \log|1 + e^x| + C$

$= e^x - \log|1 + e^x| + C'$, where $C' = C + 1$

7. Let I

$$= \int\left\{(2x-3)^5 + \frac{1}{(7x-5)^3} + \frac{1}{\sqrt{5x-4}} + \frac{1}{2-3x} + \sqrt{3x+2}\right\}dx$$

Then, $I = \int (2x-3)^5 dx + \int (7x-5)^{-3} dx$

$$+ \int (5x-4)^{-1/2} dx + \int \frac{1}{2-3x}dx + \int \sqrt{3x+2}\,dx$$

$$\Rightarrow I = \frac{(2x-3)^6}{2\times 6} + \frac{(7x-5)^{-2}}{7\times -2} + \frac{(5x-4)^{1/2}}{5\times\frac{1}{2}}$$

$$+ \left(\frac{1}{-3}\right)\log|2-3x| + \frac{(3x+2)^{3/2}}{3\times\frac{3}{2}} + C$$

$$\Rightarrow I = \frac{1}{12}(2x-3)^6 - \frac{1}{14}(7x-5)^{-2} + \frac{2}{5}\sqrt{5x-4}$$

$$- \frac{1}{3}\log|2-3x| + \frac{2}{9}(3x+2)^{3/2} + C$$

8. Let $I = \int \frac{1}{\sqrt{1-2x}+\sqrt{3-2x}}dx$. Then,

$$\Rightarrow\ I = \int \frac{\sqrt{1-2x}-\sqrt{3-2x}}{(\sqrt{1-2x}+\sqrt{3-2x})(\sqrt{1-2x}-\sqrt{3-2x})}dx$$

$$\Rightarrow\ I = \int \frac{\sqrt{1-2x}-\sqrt{3-2x}}{(1-2x)-(3-2x)}dx$$

$$\Rightarrow\ I = -\frac{1}{2}\int\sqrt{1-2x}\,dx + \frac{1}{2}\int\sqrt{3-2x}\,dx$$

$$\Rightarrow\ I = -\frac{1}{2}\left\{\frac{(1-2x)^{3/2}}{-2\times\frac{3}{2}}\right\} + \frac{1}{2}\left\{\frac{(3-2x)^{3/2}}{-2\times\frac{3}{2}}\right\} + C$$

$$= \frac{1}{6}(1-2x)^{3/2} - \frac{1}{6}(3-2x)^{3/2} + C$$

9. (i) Let $I = \int \cos^3 x\,dx$. Then,

$$I = \int \frac{\cos 3x + 3\cos x}{4}dx = \frac{1}{4}\int(\cos 3x + 3\cos x)\,dx$$

$$= \frac{1}{4}\left\{\frac{\sin 3x}{3} + 3\sin x\right\} + c$$

(ii) Let $I = \int \sin^2 x\cos^2 x\,dx$. Then,

$$I = \frac{1}{4}\int(2\sin x\cos x)^2\,dx$$

$$\Rightarrow I = \frac{1}{4}\int\sin^2 2x\,dx = \frac{1}{4}\int\frac{1-\cos 4x}{2}dx$$

$$= \frac{1}{8}\int(1-\cos 4x)dx = \frac{1}{8}\left\{x - \frac{\sin 4x}{4}\right\} + c$$

(iii) Let $I = \int \sin 4x\cos 3x\,dx$. Then,

$$I = \frac{1}{2}\int 2\sin 4x\cos 3x\,dx = \frac{1}{2}\int(\sin 7x + \sin x)dx$$

$$= \frac{1}{2}\left\{-\frac{\cos 7x}{7} - \cos x\right\} + c$$

10. Let $I = \int \frac{dx}{5\cos x - 12\sin x}$

Put $5 = r\sin\alpha$, $12 = r\cos\alpha$

$\therefore$ $5\cos x - 12\sin x = r\sin\alpha\cos x - r\cos\alpha\sin x$

$= r\sin(\alpha - x) = -r\sin(x - \alpha)$

Where $r = \sqrt{5^2+12^2} = 13,\ \alpha = \tan^{-1}\left(\frac{5}{12}\right)$

Now $I = -\frac{1}{13}\int\frac{dx}{\sin(x-\alpha)}$

$$= -\frac{1}{13}\int\text{cosec}\,(x-\alpha)\,dx$$

$$= -\frac{1}{13}\log|\text{cosec}\,(x-\alpha) - \cot\,(x-\alpha)| + c,$$

where, $\alpha = \tan^{-1}\left(\frac{5}{12}\right)$

11. (i) Let $I = \int\frac{1}{16-9x^2}dx$. Then,

$$I = \frac{1}{9}\int\frac{1}{\frac{16}{9}-x^2}dx = \frac{1}{9}\int\frac{1}{\left(\frac{4}{3}\right)^2 - x^2}dx$$

$$\Rightarrow I = \frac{1}{9}\times\frac{1}{2\left(\frac{4}{3}\right)}\times\log\left|\frac{\frac{4}{3}+x}{\frac{4}{3}-x}\right| + C = \frac{1}{24}\log\left|\frac{4+3x}{4-3x}\right| + C$$

(ii) Let $I=\int\frac{dx}{\sqrt{1-e^{2x}}}$

Put $\sqrt{1-e^{2x}}=t\Rightarrow 1-e^{2x}=t^2\Rightarrow -e^{2x}\cdot 2dx=2t\,dt$

$$\Rightarrow dx=\frac{tdt}{-e^{2x}}=\frac{t}{t^2-1}dt$$

$$\therefore\ I=\int\frac{1}{t}\cdot\frac{t}{t^2-1}dt=\int\frac{1}{t^2-1^2}\,dt$$

$$=\frac{1}{2\cdot 1}\log\left|\frac{t-1}{t+1}\right|+C=\frac{1}{2}\log\left|\frac{\sqrt{1-e^{2x}}-1}{\sqrt{1-e^{2x}}+1}\right|+C$$

12. (i) $\int\sqrt{\frac{1-x}{1+x}}\,dx=\int\sqrt{\frac{1-x}{1+x}\times\frac{1-x}{1-x}}\,dx=\int\frac{1-x}{\sqrt{1-x^2}}dx$

$$=\int\frac{1}{\sqrt{1-x^2}}dx-\int\frac{x}{\sqrt{1-x^2}}dx=I_1-I_2$$

where $I_1=\int\frac{1}{\sqrt{1-x^2}}dx=\int\frac{1}{\sqrt{1^2-x^2}}dx=\sin^{-1}x+C_1$

and $I_2=\int\frac{x}{\sqrt{1-x^2}}dx$

$$\left(\text{Put } 1-x^2=t\Rightarrow -2x\,dx=dt\Rightarrow x\,dx=-\frac{1}{2}dt\right)$$

$$\Rightarrow I_2=-\frac{1}{2}\int\frac{dt}{\sqrt{t}}=-\frac{1}{2}\cdot\frac{t^{1/2}}{\frac{1}{2}}+C_2=-\sqrt{1-x^2}+C_2$$

$\therefore\ \int\sqrt{\frac{1-x}{1+x}}\,dx=\sin^{-1}x+\sqrt{1-x^2}+C$, where $C=C_1+C_2$.

(ii) Dividing the numerator and denominator by $\cos^2 x$, we get

$$\int\frac{dx}{a^2\sin^2 x+b^2\cos^2 x}=\int\frac{\sec^2 x}{a^2\tan^2 x+b^2}dx$$

(Put $\tan x=t\Rightarrow\sec^2 x\,dx=dt$)

$$=\int\frac{dt}{a^2t^2+b^2}=\frac{1}{a^2}\int\frac{dt}{t^2+\left(\frac{b}{a}\right)^2}$$

$$=\frac{1}{a^2}\cdot\frac{1}{b/a}\tan^{-1}\left(\frac{t}{b/a}\right)+C=\frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\tan x\right)+C.$$

13. Let $I=\int\frac{1}{2x^2+x-1}dx.$

$$\Rightarrow I=\frac{1}{2}\int\frac{1}{x^2+\frac{x}{2}-\frac{1}{2}}dx$$

$$\Rightarrow I=\frac{1}{2}\int\frac{1}{x^2+x/2+(1/4)^2-(1/4)^2-1/2}dx$$

$$\Rightarrow I=\frac{1}{2}\int\frac{1}{(x+1/4)^2-(3/4)^2}dx$$

$$\Rightarrow I=\frac{1}{2}\times\frac{1}{2(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right|+C$$

$$=\frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right|+C=\frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right|+C$$

14. Let $I=\int\frac{1}{\sqrt{9+8x-x^2}}dx.$

$$\Rightarrow I=\int\frac{1}{\sqrt{-(x^2-8x-9)}}dx$$

$$\Rightarrow I=\int\frac{1}{\sqrt{-(x^2-8x+16-25)}}dx$$

$$\Rightarrow I=\int\frac{1}{\sqrt{-\{(x-4)^2-5^2\}}}dx=\int\frac{1}{\sqrt{5^2-(x-4)^2}}dx$$

$$=\sin^{-1}\left(\frac{x-4}{5}\right)+C.$$

15. Let $I=\int\frac{\cos x}{\sqrt{\sin^2 x-2\sin x-3}}\,dx$

Put $\sin x=t\Rightarrow\cos x\,dx=dt$

$$\therefore\quad I=\int\frac{dt}{\sqrt{t^2-2t-3}}=\int\frac{dt}{\sqrt{t^2-2t+1-1-3}}$$

$$\Rightarrow\quad I=\int\frac{dt}{\sqrt{(t-1)^2-2^2}}$$

$$\Rightarrow\quad I=\log|(t-1)+\sqrt{(t-1)^2-2^2}|+c$$

$$\Rightarrow\quad I=\log|t-1+\sqrt{t^2-2t-3}|+c$$

$$\Rightarrow\quad I=\log\left|(\sin x-1)+\sqrt{\sin^2 x-2\sin x-3}\right|+c$$

16. Let $I=\int\frac{2x+3}{2x^2+3x-1}\,dx$

Let $2x+3=A\left[\frac{d}{dx}(2x^2+3x-1)\right]+B=A(4x+3)+B$

$\Rightarrow\quad 2x+3=4Ax+(3A+B)$

Comparing the coefficient of x and constant on both sides, we get $4A=2$ and $3A+B=3$

$\Rightarrow\quad A=\frac{1}{2}$ and $3\left(\frac{1}{2}\right)+B=3\Rightarrow B=\frac{3}{2}$

$$\Rightarrow\quad 2x+3=\frac{1}{2}(4x+3)+\frac{3}{2}$$

$$\therefore\quad I=\int\frac{\frac{1}{2}(4x+3)+\frac{3}{2}}{2x^2+3x-1}\,dx$$

$$=\frac{1}{2}\int\frac{4x+3}{2x^2+3x-1}dx+\frac{3}{2}\int\frac{1}{2x^2+3x-1}dx$$

$$=\frac{1}{2}I_1+\frac{3}{2}I_2$$

I_1 is of the type $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$

$\Rightarrow \quad I_1 = \log|2x^2+3x-1| + c_1$

$I_2 = \int \frac{1}{2x^2+3x-1}dx = \frac{1}{2}\int \frac{1}{x^2+\frac{3}{2}x-\frac{1}{2}}dx$

$= \frac{1}{2}\int \frac{1}{\left(x^2+\frac{3}{2}x+\frac{9}{16}\right)-\frac{9}{16}-\frac{1}{2}}dx$

$= \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{4}\right)^2-\left(\frac{\sqrt{17}}{4}\right)^2}dx$

$= \frac{1}{2}\times\frac{1}{2\times\frac{\sqrt{17}}{4}}\log\left|\frac{x+\frac{3}{4}-\frac{\sqrt{17}}{4}}{x+\frac{3}{4}+\frac{\sqrt{17}}{4}}\right| + c_2$

$= \frac{1}{\sqrt{17}}\log\left|\frac{4x+3-\sqrt{17}}{4x+3+\sqrt{17}}\right| + c_2$

$\Rightarrow \quad I = \frac{1}{2}\log|2x^2+3x-1| + \frac{3}{2\sqrt{17}}\log\left|\frac{4x+3-\sqrt{17}}{4x+3+\sqrt{17}}\right| + c,$

where $c = c_1 + c_2$

17. We have, $\frac{3x+2}{x^3-6x^2+11x-6} = \frac{3x+2}{(x-1)(x-2)(x-3)}$

Let $\frac{3x+2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$.

$\Rightarrow \quad 3x+2 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$ ...(i)

Putting $x - 1 = 0$ or, $x = 1$ in (i), we get $5 = A(1-2)(1-3)$

$\Rightarrow A = \frac{5}{2}$

Putting $x - 2 = 0$ or, $x = 2$ in (i), we obtain
$8 = B(2-1)(2-3)$
$\Rightarrow B = -8$
Putting $x - 3 = 0$ or, $x = 3$ in (i), we obtain

$11 = C(3-1)(3-2) \Rightarrow C = \frac{11}{2}$

$\therefore \quad \frac{3x+2}{x^3-6x^2+11x-6} = \frac{3x+2}{(x-1)(x-2)(x-3)}$

$= \frac{5}{2(x-1)} - \frac{8}{x-2} + \frac{11}{2(x-3)}$

18. We have, $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)}$

So, let $\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$. Then,

$2x = A(x^2+x+1) + (Bx+C)(x-1)$ ...(i)

Putting $x - 1 = 0$ or, $x = 1$ in (i), we get

$2 = 3A \Rightarrow A = \frac{2}{3}$

Putting $x = 0$ in (i), we get

$A - C = 0 \Rightarrow C = A = \frac{2}{3}$

Putting $x = -1$ in (i), we get

$-2 = A + 2B - 2C$

$\Rightarrow -2 = \frac{2}{3} + 2B - \frac{4}{3} \Rightarrow B = -\frac{2}{3}$

$\therefore \quad \frac{2x}{x^3-1} = \frac{2}{3}\cdot\frac{1}{x-1} + \frac{-(2/3)x+(2/3)}{x^2+x+1}$

$\Rightarrow \quad \frac{2x}{x^3-1} = \frac{2}{3}\cdot\frac{1}{x-1} + \frac{2}{3}\cdot\frac{(-x+1)}{x^2+x+1}$

19. Let $I = \int \frac{1-\cos x}{\cos x(1+\cos x)}dx$ and let $\cos x = y$. Then,

$\frac{1-\cos x}{\cos x(1+\cos x)} = \frac{1-y}{y(1+y)}$

Let $\frac{1-y}{y(1+y)} = \frac{A}{y} + \frac{B}{1+y}$ ...(i)

$\Rightarrow \quad 1 - y = A(1+y) + By$ ...(ii)

Putting $y = 0$ in (ii), we get $A = 1$.

Putting $y = -1$ in (ii), we get $B = -2$.

Substituting the values of A and B in (i), we get

$\frac{1-y}{y(1+y)} = \frac{1}{y} - \frac{2}{1+y}$

$\Rightarrow \frac{1-\cos x}{\cos x(1+\cos x)} = \frac{1}{\cos x} - \frac{2}{1+\cos x}$ $[\because y = \cos x]$

$\therefore \quad I = \int \frac{1}{\cos x}dx - \int \frac{2}{1+\cos x}dx$

$\Rightarrow I = \int \sec x\, dx - \int \frac{2}{2\cos^2(x/2)}dx = \int \sec x\, dx - \int \sec^2(x/2)\, dx$

$\Rightarrow \quad I = \log|\sec x + \tan x| - 2\tan(x/2) + C$

20. Let $x^2 = y$. Then,

$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$ ...(i)

Let $\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$ ...(ii)

$\Rightarrow \quad y = A(y+4) + B(y+1)$

Putting $y = -1$ and $y = -4$ successively in (ii), we get

$A = -\frac{1}{3}$ and $B = \frac{4}{3}$.

Substituting the values of A and B in (ii), we get

$\frac{y}{(y+1)(y+4)} = -\frac{1}{3(y+1)} + \frac{4}{3(y+4)}$

Replacing y by x^2, we obtain

$$\frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

$$\therefore I = \int \frac{x^2}{(x^2+1)(x^2+4)}dx = \int\left\{-\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}\right\}dx$$

$$= -\frac{1}{3}\int\frac{1}{x^2+1}dx + \frac{4}{3}\int\frac{1}{x^2+4}dx$$

$$\Rightarrow I = -\frac{1}{3}\tan^{-1}x + \frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right)+C$$

21. Let $I = \int \log(1+x^2)dx$

$$I = \int \underset{\text{I}}{\log(1+x^2)}.\underset{\text{II}}{1}\,dx = x\log(1+x^2) - \int\left(\frac{1}{1+x^2}.2x\right)x\,dx$$

$$\Rightarrow I = x\log(1+x^2) - 2\int\frac{x^2}{x^2+1}dx$$

$$= x\log(x^2+1) - 2\int\frac{x^2+1-1}{x^2+1}dx$$

$$\Rightarrow I = x\log(x^2+1) - 2\int\left(1-\frac{1}{x^2+1}\right)dx$$

$$= x\log(x^2+1) - 2(x - \tan^{-1}x) + C$$

$$\Rightarrow \quad I = x\log(x^2+1) - 2x + 2\tan^{-1}x + C$$

22. Let $I = \int \underset{\text{I}}{x^3}\,\underset{\text{II}}{e^x}\,dx$. Then,

$$I = x^3e^x - \int 3x^2e^x dx = x^3e^x - 3\int \underset{\text{I}}{x^2}\,\underset{\text{II}}{e^x}\,dx$$

$$\Rightarrow I = x^3e^x - 3\left\{x^2e^x - \int 2xe^x\,dx\right\}$$

$$= x^3e^x - 3\left\{x^2e^x - 2\int \underset{\text{I}}{x}\,\underset{\text{II}}{e^x}\,dx\right\}$$

$$\Rightarrow I = x^3e^x - 3\{x^2e^x - 2(xe^x - e^x)\} + C$$

$$\Rightarrow \quad I = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = (x^3 - 3x^2 + 6x - 6)e^x + C$$

23. Let $I = \int \frac{x - \sin x}{1 - \cos x}dx$

$$= \int \frac{x - \sin x}{2\sin^2 x/2}dx$$

$$= \frac{1}{2}\int x\text{cosec}^2\frac{x}{2}dx - \int\frac{2\sin x/2\cos x/2}{2\sin^2 x/2}dx$$

$$= \frac{1}{2}I_1 - \int\cot\frac{x}{2}dx \qquad \text{...(i)}$$

Where $I_1 = \int x\,\text{cosec}^2\frac{x}{2}dx$

$$= x\int\text{cosec}^2\frac{x}{2}dx + 2\int 1\cdot\cot\frac{x}{2}dx$$

$$= -2x\cot\frac{x}{2} + 2\int\cot\frac{x}{2}dx + C$$

From (i), we have

$$I = \frac{1}{2}\left[-2x\cot\frac{x}{2} + 2\int\cot\frac{x}{2}dx\right] - \int\cot\frac{x}{2}dx + C$$

$$= -x\cot\frac{x}{2} + \int\cot\frac{x}{2}dx - \int\cot\frac{x}{2}dx + C$$

$$= -x\cot\frac{x}{2} + C$$

24. Let $I = \int\frac{x-1}{(x+1)^3}e^x dx$

$$= \int\frac{x+1-2}{(x+1)^3}e^x dx$$

$$= \int e^x\left[\frac{1}{(x+1)^2} + \frac{(-2)}{(x+1)^3}\right]dx$$

$$= \int e^x\left[\frac{1}{(x+1)^2} + \frac{d}{dx}\left(\frac{1}{(x+1)^2}\right)\right]dx$$

$$= \frac{e^x}{(x+1)^2} + C \qquad \left[\because \int e^x(f(x) + f'(x))\,dx = e^x f(x) + C\right]$$

25. Let $I = \int\left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dx$

Put $u = \log x \Rightarrow x = e^u \Rightarrow dx = e^u du$

$$\therefore \quad I = \int\left[\frac{1}{u} - \frac{1}{u^2}\right]e^u du$$

$$= \int e^u\left[\frac{1}{u} + \frac{d}{dx}\left(\frac{1}{u}\right)\right]dx$$

$$= \frac{e^u}{u} + C \qquad \left[\because \int e^x(f(x) + f'(x))\,dx = e^x f(x) + C\right]$$

$$= \frac{e^{\log x}}{\log x} + C = \frac{x}{\log x} + C$$

26. Let $I = \int\cos x\sqrt{4 + \sin^2 x}\,dx$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\therefore \quad I = \int\sqrt{2^2 + t^2}\,dt$$

$$= \frac{t\sqrt{2^2+t^2}}{2} + \frac{2^2}{2}\log\left|t + \sqrt{2^2+t^2}\right| + C$$

$$= \frac{\sin x\sqrt{4 + \sin^2 x}}{2} + 2\log\left|\sin x + \sqrt{4 + \sin^2 x}\right| + C.$$

27. Let $I = \int\sin^{-1}\sqrt{x}\,dx$

Put $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\,dt$

$$\therefore \; I = \int\sin^{-1}t\cdot 2t dt = 2\int t\sin^{-1}t\,dt$$

$$= 2\left[\sin^{-1}t\cdot\frac{t^2}{2} - \int\frac{1}{\sqrt{1-t^2}}\cdot\frac{t^2}{2}dt\right]$$

$$= 2\left[\frac{t^2}{2}\sin^{-1}t - \frac{1}{2}\int\frac{-(1-t^2)+1}{\sqrt{1-t^2}}dt\right]$$

$$= 2\left[\frac{t^2}{2}\sin^{-1}t + \frac{1}{2}\int\sqrt{1-t^2}\,dt - \frac{1}{2}\int\frac{1}{\sqrt{1-t^2}}\,dt\right]$$

$$= 2\left[\frac{t^2}{2}\sin^{-1}t + \frac{1}{2}\left[\frac{t\sqrt{1-t^2}}{2} + \frac{1}{2}\sin^{-1}t\right] - \frac{1}{2}\sin^{-1}t + C_1\right]$$

$$= t^2\sin^{-1}t + \frac{t\sqrt{1-t^2}}{2} + \frac{1}{2}\sin^{-1}t - \sin^{-1}t + C$$

$$= \frac{(2t^2-1)}{2}\sin^{-1}t + \frac{t\sqrt{1-t^2}}{2} + C$$

$$= \frac{1}{2}\left((2x-1)\sin^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x}\right) + C.$$

28. $\int\sqrt{1+2x-3x^2}\,dx = \sqrt{3}\int\sqrt{\frac{1}{3}+\frac{2}{3}x-x^2}\,dx$

$$= \sqrt{3}\int\sqrt{\frac{1}{3}-\left(x^2-\frac{2}{3}x+\frac{1}{9}\right)+\frac{1}{9}}\,dx$$

$$= \sqrt{3}\int\sqrt{\left(\frac{2}{3}\right)^2-\left(x-\frac{1}{3}\right)^2}\,dx$$

$$= \frac{\sqrt{3}}{2}\left(x-\frac{1}{3}\right)\sqrt{\frac{4}{9}-\left(x-\frac{1}{3}\right)^2} + \frac{2\sqrt{3}}{9}\sin^{-1}\frac{3}{2}\left(x-\frac{1}{3}\right)+C$$

$$= \frac{\sqrt{3}}{2}\frac{(3x-1)}{3}\sqrt{\frac{1}{3}+\frac{2}{3}x-x^2} + \frac{2\sqrt{3}}{9}\sin^{-1}\left(\frac{3x-1}{2}\right)+C$$

$$= \frac{(3x-1)}{6}\sqrt{1+2x-3x^2} + \frac{2\sqrt{3}}{9}\sin^{-1}\left(\frac{3x-1}{2}\right)+C.$$

29. $\int_3^5\frac{dt}{1+3t} = \left[\frac{\log(1+3t)}{3}\right]_3^5$

$$= \frac{1}{3}(\log 16 - \log 10)$$

$$= \frac{1}{3}\log\frac{16}{10} = \frac{1}{3}\log\frac{8}{5}$$

30. $\int_1^2\frac{2}{4x^2-1}\,dx = \frac{2}{4}\int_1^2\frac{dx}{x^2-\frac{1}{4}}$

$$= \frac{1}{2}\int_1^2\frac{dx}{x^2-\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2}\left[\frac{1}{2(1/2)}\log\left|\frac{x-1/2}{x+1/2}\right|\right]_1^2$$

$$= \frac{1}{2}\left[\log\left|\frac{2x-1}{2x+1}\right|\right]_1^2 = \frac{1}{2}\left[\log\frac{3}{5}-\log\frac{1}{3}\right]$$

$$= \frac{1}{2}\log\frac{9}{5}$$

31. $\int_0^1 x^2e^x\,dx = [x^2\cdot e^x]_0^1 - \int_0^1 2xe^x\,dx$

$$= (1.e^1-0) - 2\int_0^1 xe^x\,dx = e - 2\left([xe^x]_0^1 - \int_0^1 1.e^x\,dx\right)$$

$$= e - 2(1\cdot e^1 - 0) + 2\int_0^1 e^x\,dx = e - 2e + 2[e^x]_0^1$$

$= -e + 2(e^1 - e^0) = -e + 2(e-1) = e - 2.$

32. $\int_0^{\sqrt{2}}\sqrt{2-x^2}\,dx = \int_0^{\sqrt{2}}\sqrt{(\sqrt{2})^2-x^2}\,dx$

$$= \left[\frac{x}{2}\sqrt{2-x^2}+\frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^{\sqrt{2}}$$

$=[0 + \sin^{-1}1 - 0 - \sin^{-1}0]$

$= \frac{\pi}{2}$

33. Let $I = \int_0^4\frac{1}{x+\sqrt{x}}\,dx$

Put $x = t^2$. Then, $dx = 2t\,dt$.
When $x = 0$, $t^2 = 0 \Rightarrow t = 0$. When $x = 4 \Rightarrow t^2 = 4 \Rightarrow t = 2$

$$\therefore\ I = \int_0^2\frac{2t\,dt}{t^2+t} = 2\int_0^2\frac{1}{t+1}\,dt$$

$$\Rightarrow\ I = 2[\log(t+1)]_0^2 = 2[\log 3 - \log 1] = 2\log 3$$

34. Let $I = \int_0^{\pi/2}\frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}\,d\theta$

Put $\sin\theta = t \Rightarrow \cos\theta\,d\theta = dt$
When $\theta = 0$, $t = \sin 0 = 0$

When $\theta = \frac{\pi}{2}$, $t = \sin\frac{\pi}{2} = 1$

$$\therefore\ I = \int_0^1\frac{dt}{(1+t)(2+t)} = \int_0^1\left(\frac{1}{1+t}-\frac{1}{2+t}\right)dt$$

(Using partial fraction)

$$\Rightarrow\ I = [\log(1+t)-\log(2+t)]_0^1$$

$= \log 2 - \log 3 - (\log 1 - \log 2)$

$= 2\log 2 - \log 3 = \log\frac{4}{3}$

35. Let $I = \int_0^{1/\sqrt{2}}\frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$

Put $\sin^{-1}x = \theta$ or, $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$
When, $x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

and when $x = \frac{1}{\sqrt{2}} \Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore\ I = \int_0^{\pi/4}\frac{\theta}{\cos^3\theta}\cos\theta\,d\theta = \int_0^{\pi/4}\underset{\text{I}}{\theta}\,\underset{\text{II}}{\sec^2\theta}\,d\theta$$

$$= [\theta\tan\theta]_0^{\pi/4} - \int_0^{\pi/4}1\times\tan\theta\,d\theta$$

$$\Rightarrow I = [\theta\tan\theta]_0^{\pi/4} + [\log|\cos\theta|]_0^{\pi/4}$$

$$= \left(\frac{\pi}{4}-0\right) + \left\{\log\left(\frac{1}{\sqrt{2}}\right)-\log 1\right\} = \frac{\pi}{4}-\frac{1}{2}\log 2$$

36. Let $I = \int_0^{\pi/2}\{\sqrt{\tan x}+\sqrt{\cot x}\}\,dx.$

$$\Rightarrow\ I = \int_0^{\pi/2}\left\{\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right\}dx = \int_0^{\pi/2}\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}\,dx$$

$$= \sqrt{2}\int_0^{\pi/2}\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}\,dx$$

$$\Rightarrow I = \sqrt{2}\int_0^{\pi/2}\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}\,dx$$

Let $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$
When, $x = 0 \Rightarrow t = \sin 0 - \cos 0 = -1$

and when $x = \frac{\pi}{2} \Rightarrow t = \sin\frac{\pi}{2} - \cos\frac{\pi}{2} = 1$

$$\therefore I = \sqrt{2}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\,[\sin^{-1} t]_{-1}^{1}$$

$$\Rightarrow I = \sqrt{2}\left\{\sin^{-1}1 - \sin^{-1}(-1)\right\}$$

$$= \sqrt{2}\left\{2\sin^{-1}(1)\right\} = 2\sqrt{2}\left(\frac{\pi}{2}\right) = \sqrt{2}\pi$$

37. Given $f(x) = |x| + |x+2| + |x+5|$
or $f(x) = |x+5| + |x+2| + |x|$
When $-5 \le x \le -2$, $x+5 \ge 0$, $x+2 \le 0$, $x \le 0$
$\Rightarrow$ $|x+5| = x+5$, $|x+2| = -(x+2)$, $|x| = -x$
$\Rightarrow$ $f(x) = (x+5) - (x+2) - x = -x+3$
and when $-2 \le x \le 0$, $x+5 \ge 0$, $x+2 \ge 0$, $x \le 0$
$\Rightarrow$ $|x+5| = x+5$, $|x+2| = x+2$, $|x| = -x$
$\Rightarrow$ $f(x) = (x+5) + (x+2) - x = x+7$.

$$\therefore \int_{-5}^{0} f(x)dx = \int_{-5}^{-2} f(x)\,dx + \int_{-2}^{0} f(x)dx$$

$$= \int_{-5}^{-2}(-x+3)dx + \int_{-2}^{0}(x+7)dx = \left[-\frac{x^2}{2}+3x\right]_{-5}^{-2} + \left[\frac{x^2}{2}+7x\right]_{-2}^{0}$$

$$= -\frac{1}{2}(4-25) + 3(-2+5) + \frac{1}{2}(0-4) + 7(0+2)$$

$$= \frac{21}{2} + 9 - 2 + 14 = \frac{63}{2} = 31\frac{1}{2}.$$

38. First note that the given function is discontinuous at $x = 1$.

$$\therefore \int_{-1}^{2} f(x)dx = \int_{-1}^{1} f(x)dx + \int_{1}^{2} f(x)dx$$

$$\left(\because \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right)$$

$$= \int_{-1}^{1}(2x+1)dx + \int_{1}^{2}(x-5)dx$$

$$= [x^2+x]_{-1}^{1} + \left[\frac{x^2}{2} - 5x\right]_1^2$$

$$= (1+1) - (1-1) + (2-10) - \left(\frac{1}{2} - 5\right) = 2 - 0 - 8 + \frac{9}{2} = -\frac{3}{2}.$$

39. Let $f(x) = x^3 \sin^4 x \Rightarrow f(-x) = (-x)^3 \sin^4(-x) = -x^3 \sin^4 x$ $= -f(x)$

$\Rightarrow$ $f(x)$ is on odd function.

$$\therefore \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx = 0$$

$$\left(\because \int_{-a}^{a} f(x)dx = 0, \text{ if } f(-x) = -f(x)\right)$$

40. Let $I = \int_0^{\pi}\frac{x\tan x}{\sec x + \tan x}dx = \int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$ ...(1)

Then, by using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, we get

$$I = \int_0^{\pi}\frac{(\pi - x)\sin(\pi - x)}{1+\sin(\pi - x)}dx = \int_0^{\pi}\frac{(\pi - x)\sin x}{1+\sin x}dx \quad ...(2)$$

On adding (1) and (2), we get

$$2I = \int_0^{\pi}\frac{(x+\pi-x)\sin x}{1+\sin x}dx = \pi\int_0^{\pi}\frac{\sin x}{1+\sin x}dx$$

$$= \pi\int_0^{\pi}\left(1 - \frac{1}{1+\sin x}\right)dx$$

$$= \pi\int_0^{\pi}1\,dx - \pi\int_0^{\pi}\frac{dx}{1+\cos\left(\frac{\pi}{2}-x\right)}$$

$$= \pi[x]_0^{\pi} - \pi\int_0^{\pi}\frac{dx}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}$$

$$= \pi[\pi - 0] - \frac{\pi}{2}\int_0^{\pi}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx = \pi^2 - \frac{\pi}{2}\left[\frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}{-\frac{1}{2}}\right]_0^{\pi}$$

$$= \pi^2 + \pi\left[\tan\left(-\frac{\pi}{4}\right) - \tan\frac{\pi}{4}\right] = \pi^2 + \pi(-1-1) = \pi(\pi - 2)$$

$$\Rightarrow I = \pi\left(\frac{\pi}{2} - 1\right).$$

41. Let $I = \int_0^{\pi} x f(\sin x)dx$ ...(i)

$$= \int_0^{\pi}(\pi - x) f(\sin(\pi - x))dx \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\pi}(\pi - x) f(\sin x)dx \quad ...(ii)$$

On adding (i) and (ii), we get

$$2I = \int_0^{\pi}\pi f(\sin x)dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$