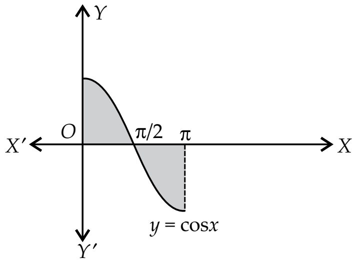
CHAPTER 8

Application of Integrals

EXAM DRILL

SOLUTIONS

1. (a) : We have given the equations $y = \cos x$, $x = 0$ and $x = \pi$

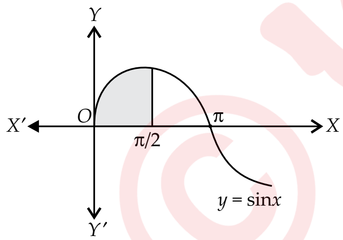


Thus the required area $= \int_0^{\pi/2} \cos x\,dx + \left| \int_{\pi/2}^{\pi} \cos x\,dx \right|$

$= \left[\sin\frac{\pi}{2} - \sin 0 \right] + \left| \sin\pi - \sin\frac{\pi}{2} \right|$

$= 1 + 1 = 2$ sq. units

2. (d) : We have given the equations $y = \sin x$, the abscissa $x = 0$, $x = \pi/2$ and X-axis *i.e.*, $y = 0$

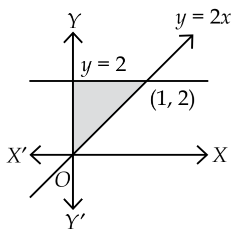


Thus the required area $= \int_0^{\pi/2} \sin x\,dx$

$= -[\cos x]_0^{\pi/2} = -\left[\cos\frac{\pi}{2} - \cos 0 \right]$

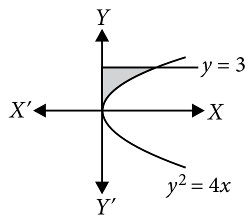
$= -[0 - 1] = 1$ sq. unit

3. (a) : Given line is $y = 2x$



$\therefore$ Required area $= \int_0^2 \frac{y}{2}\,dy = \left[\frac{y^2}{4} \right]_0^2 = \frac{4}{4} = 1$ sq. unit

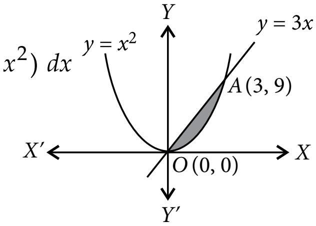
4. (b) : $\therefore$ Required area



$= \int_0^3 \left[\frac{y^2}{4} \right] dy$

$= \left[\frac{y^3}{12} \right]_0^3 = \frac{27}{12} = \frac{9}{4}$ sq. units

5. (c) : The intersection points of given curves are O (0, 0) and A (3, 9).



$\therefore$ Required area $= \int_0^3 (3x - x^2)\,dx$

$= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3$

$= \frac{27}{6} = 4.5$ sq. units

6. (b) : For minima put $y'(x) = 0$

$\Rightarrow \quad 4x^3 - 6x^2 + 2x = 0$

$\Rightarrow \quad 2x(2x^2 - 3x + 1) = 0$

$\Rightarrow \quad x = 0, 1, \frac{1}{2}$

Now, $y''(x) = 12x^2 - 12x + 2$

$y''(0) = 2 > 0$

$y''(1) = 12 - 12 + 2 = 2 > 0$

$y''\left(\frac{1}{2}\right) = \frac{12}{4} - \frac{12}{2} + 2 = 3 - 6 + 2 = -1 < 0$

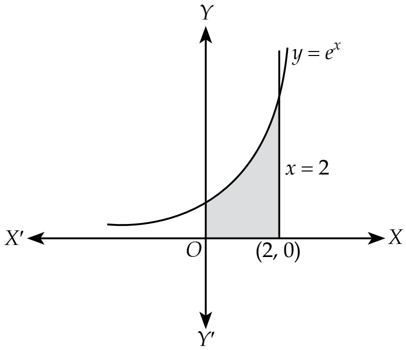
Thus, $x = 0$ and $x = 1$ are the required ordinates.

$\therefore$ Required area $= \int_0^1 (x^4 - 2x^3 + x^2 + 3)\,dx$

$= \left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} + 3x \right]_0^1$

$= \frac{1}{5} - \frac{1}{2} + \frac{1}{3} + 3 = \frac{6 - 15 + 10 + 90}{30} = \frac{91}{30}$ sq. units

7.



$\therefore$ Required area $= \int_0^2 y\,dx = \int_0^2 e^x dx = [e^x]_0^2$

$= e^2 - e^0 = (e^2 - 1)$sq. units

8. Required area $= \int_0^1 x dy = \int_0^1 y dy = \left[\frac{y^2}{2}\right]_0^1$

$= \frac{1}{2}$ sq.unit

9. **(i) (c) :** We have, $x^2 + y^2 = 16$ ...(i)

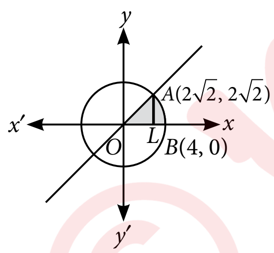
and $y = x$ ...(ii)

From (i) and (ii), $2x^2 = 16 \Rightarrow x^2 = 8 \Rightarrow x = 2\sqrt{2}$

($\because$ x lies in first quadrant)

$\therefore$ Point of intersection of (i) and (ii) in first quadrant is $(2\sqrt{2}, 2\sqrt{2})$.

(ii) (b) : The shaded region which represent the area bounded by two given curves in first quadrant is shown below.



(iii) (d) : $\int_0^{2\sqrt{2}} x\,dx = \left[\frac{x^2}{2}\right]_0^{2\sqrt{2}} = \frac{(2\sqrt{2})^2}{2} = \frac{8}{2} = 4$

(iv) (a) : $\int_{2\sqrt{2}}^{4} \sqrt{16-x^2}\,dx = \left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\cdot\sin^{-1}\left(\frac{x}{4}\right)\right]_{2\sqrt{2}}^{4}$

$= 8\sin^{-1}(1) - 4 - 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

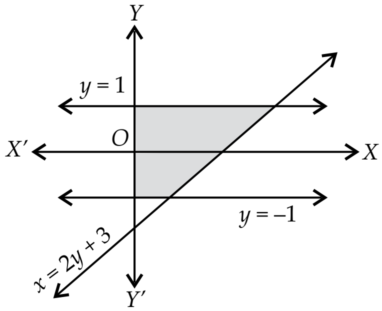
$= 8\left(\frac{\pi}{2}\right) - 4 - 8\left(\frac{\pi}{4}\right) = 4\pi - 4 - 2\pi = 2\pi - 4 = 2(\pi - 2)$

(v) (d) : Required area = Area (OLA) + Area (BAL)

$= \int_0^{2\sqrt{2}} x\,dx + \int_{2\sqrt{2}}^{4} \sqrt{16-x^2}\,dx$

$= 4 + 2(\pi - 2) = 2\pi$ sq. units.

10. We have given the equations $x = 2y + 3$, $y = 1$ and $y = -1$.

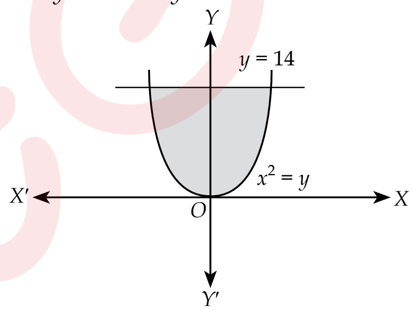


Thus the required area of the shaded region

$= \int_{-1}^{1} (2y+3)dy = \left[y^2 + 3y\right]_{-1}^{1}$

$= [1 + 3 - 1 + 3] = 6$ sq. units

11. We have $y = x^2$ and $y = 14$.



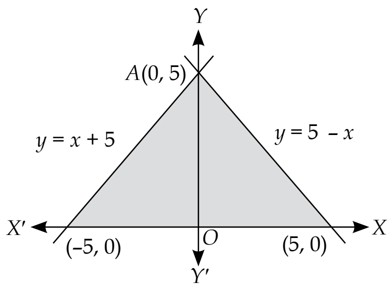
$\therefore$ Required area = Area of shaded portion

$= 2\int_0^{14} \sqrt{y}\,dy = 2\cdot\frac{2}{3}[y^{3/2}]_0^{14} = \frac{4}{3}[14^{3/2}]$ sq.units

12. We have, $y = x + 5$ ...(1)

and $y = 5 - x$ ...(2)

Point of intersection of (1) and (2) is $A(0, 5)$



$\therefore$ Required area $= \int_{-5}^{0} (x+5)dx + \int_0^5 (5-x)dx$

$= \left[\frac{x^2}{2} + 5x\right]_{-5}^{0} + \left[5x - \frac{x^2}{2}\right]_0^5 = -\left(\frac{25}{2} - 25\right) + \left(25 - \frac{25}{2}\right)$

$= \frac{-25}{2} + 25 + 25 - \frac{25}{2} = 50 - \frac{50}{2} = 25$ sq. units

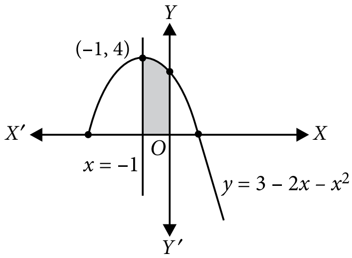
13. Given curve is $y = 3 - 2x - x^2$

$\therefore$ Required area $= \int_{-1}^{0} (3 - 2x - x^2)\,dx$

$$= \left[3x - x^2 - \frac{x^3}{3}\right]_{-1}^{0}$$

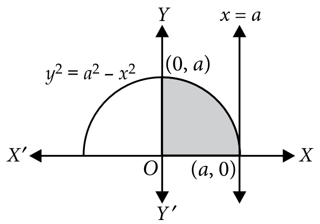
$$= (0) - \left(-3 - 1 + \frac{1}{3}\right)$$

$$= \frac{11}{3} \text{ sq. units}$$



OR

Given curve is $y = \sqrt{a^2 - x^2}$ and $x = 0$, $x = a$



$\therefore$ Required area $= \int_0^a \sqrt{a^2 - x^2}\,dx$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

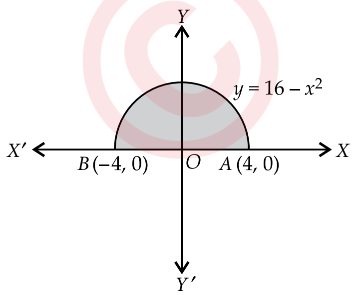
$$= \left[0 + \frac{a^2}{2}\sin^{-1}(1) - 0\right] = \frac{a^2}{2} \times \frac{\pi}{2} = \frac{\pi}{4}a^2 \text{ sq. units}$$

14. We have the equation $y = \sqrt{16 - x^2}$ and X-axis *i.e.,* $y = 0$.

$\because \quad \sqrt{16 - x^2} = 0 \Rightarrow 16 - x^2 = 0$

$\Rightarrow \quad x^2 = 16 \Rightarrow x = \pm 4$

$\therefore$ The intersecting points are $A(4, 0)$ and $B(-4, 0)$.



Thus the required area $= \int_{-4}^{4} (16 - x^2)^{1/2}\,dx$

$$= \int_{-4}^{4} \sqrt{(4^2 - x^2)}\,dx = \left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{4^2}{2}\sin^{-1}\frac{x}{4}\right]_{-4}^{4}$$

$$= \left[0 + 8\sin^{-1} 1\right] - \left[0 + 8\sin^{-1}(-1)\right]$$

$$= \left[8 \cdot \frac{\pi}{2} + 8 \cdot \frac{\pi}{2}\right] = 8\pi \text{ sq. units}$$

15. We have given the equations

$y = -x^2$ ...(1)

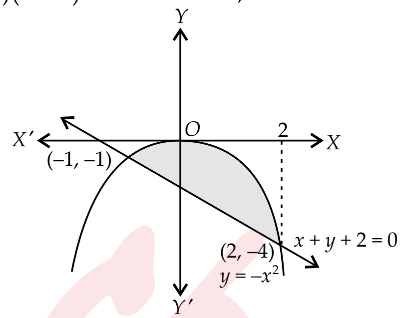
and $x + y + 2 = 0$ ...(2)

Solving (1) and (2), we get

$-x - 2 = -x^2 \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow \quad x^2 + x - 2x - 2 = 0 \Rightarrow x(x + 1) - 2(x + 1) = 0$

$\Rightarrow \quad (x - 2)(x + 1) = 0 \Rightarrow x = 2, -1$



Thus the area of the shaded region

$$= \left|\int_{-1}^{2} (-x - 2 + x^2)\,dx\right| = \left|\int_{-1}^{2} (x^2 - x - 2)\,dx\right|$$

$$= \left|\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2}\right| = \left|\frac{8}{3} - \frac{4}{2} - 4 + \frac{1}{3} + \frac{1}{2} - 2\right|$$

$$= \left|\frac{16 - 12 - 24 + 2 + 3 - 12}{6}\right| = \left|-\frac{27}{6}\right| = \frac{9}{2} \text{ sq. units}$$

OR

We have $y = x^3$, ...(1)

$y = x + 6$ ...(2)

and $x = 0$ ...(3)

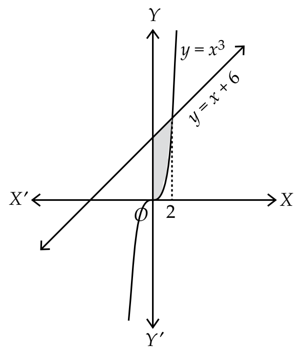
Solving (1) and (2), we get

$x^3 = x + 6$

$\Rightarrow \quad x^3 - x - 6 = 0$

$\Rightarrow \quad (x - 2)(x^2 + 2x + 3) = 0$

$\Rightarrow \quad x = 2$ with two imaginary points.



Thus the area of the shaded region $= \int_0^2 (x + 6 - x^3)\,dx$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^4}{4}\right]_0^2 = \frac{4}{2} + 12 - \frac{16}{4} - 0$$

$= 2 + 12 - 4 = 10$ sq. units

16. We have given the equations

$y = \sqrt{x}$ ...(1)

and $x = 2y + 3$ ...(2)

Solving (1) and (2), we get

$y = \sqrt{2y+3}$

Squaring on both sides, we get

$y^2 = 2y + 3$

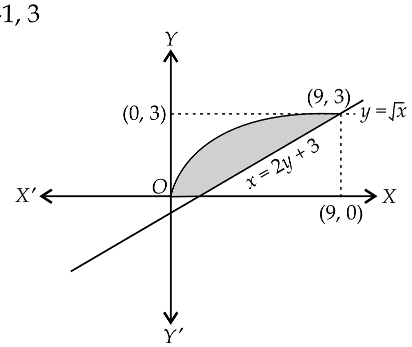
$\Rightarrow \quad y^2 - 2y - 3 = 0$

$\Rightarrow \quad y^2 - 3y + y - 3 = 0$

$\Rightarrow \quad y(y-3) + 1(y-3) = 0$

$\Rightarrow \quad (y+1)(y-3) = 0$

$\Rightarrow \quad y = -1, 3$

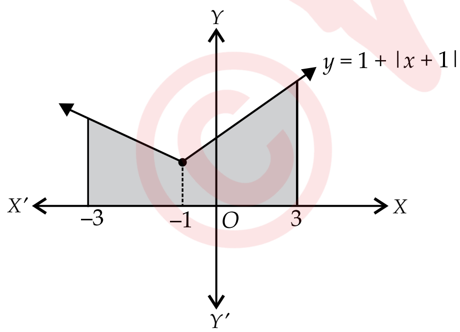


Thus the required area of the shaded region

$= \int_0^3 (2y + 3 - y^2)dy = \left[y^2 + 3y - \frac{y^3}{3}\right]_0^3$

$= [9 + 9 - 9 - 0] = 9$ sq. units

17. We have given, $y = 1 + |x + 1|$, $x = -3$, $x = 3$ and $y = 0$.



Now, $y = \begin{cases} -x, & \text{if } x < -1 \\ x + 2, & \text{if } x \ge -1 \end{cases}$

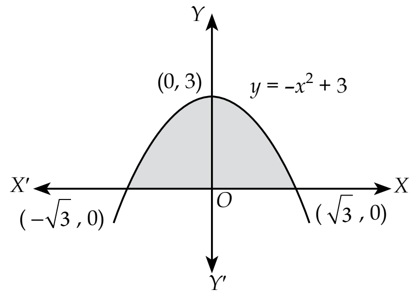
Thus the area of the shaded region

$= \int_{-3}^{-1} (-x)\,dx + \int_{-1}^{3} (x+2)dx = -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$

$= -\left[\frac{1}{2} - \frac{9}{2}\right] + \left[\frac{9}{2} + 6 - \frac{1}{2} + 2\right]$

$= 4 + 12 = 16$ sq. units

18.



$\therefore$ Required area $= \int_{-\sqrt{3}}^{\sqrt{3}} (-x^2 + 3)dx = 2\int_0^{\sqrt{3}} (-x^2 + 3)dx$

$= 2\left[\frac{-x^3}{3} + 3x\right]_0^{\sqrt{3}} = 2\left[\frac{-3\sqrt{3}}{3} + 3\sqrt{3}\right]$

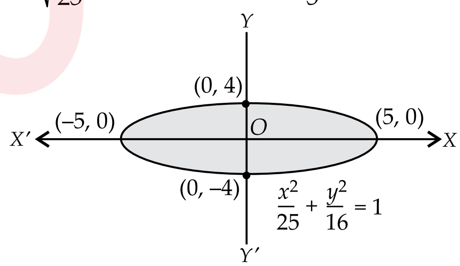
$= 2(2\sqrt{3}) = 4\sqrt{3}$ sq. units

19. We have $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Here, $a = \pm 5$ and $b = \pm 4$

and $\frac{y^2}{4^2} = 1 - \frac{x^2}{5^2} \Rightarrow y^2 = 16\left(1 - \frac{x^2}{25}\right)$

$\Rightarrow \quad y = \pm\sqrt{\frac{16}{25}(25 - x^2)} \Rightarrow y = \pm\frac{4}{5}\sqrt{(5^2 - x^2)}$



Thus the area of the shaded region

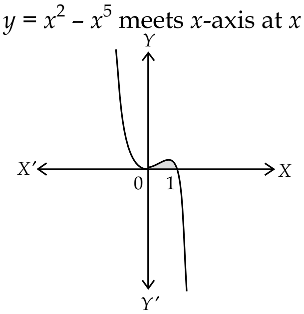
$= 2 \cdot \frac{4}{5}\int_{-5}^{5} \sqrt{5^2 - x^2}\,dx$ [$\because y$ is +ve in 1st and 2nd quadrant]

$= 2 \cdot \frac{8}{5}\int_0^5 \sqrt{5^2 - x^2}\,dx = 2 \cdot \frac{8}{5}\left[\frac{x}{2}\sqrt{5^2 - x^2} + \frac{5^2}{2}\sin^{-1}\frac{x}{5}\right]_0^5$

$= 2 \cdot \frac{8}{5}\left[0 + \frac{5^2}{2}\sin^{-1}\frac{5}{5} - 0 - 0\right]$

$= 2 \cdot \frac{8}{5}\left[\frac{25}{2} \cdot \frac{\pi}{2}\right] = 20\pi$ sq. units

20. We have, $y = x^2 - x^5$ meets x-axis at $x = 0$ and $x = 1$



Thus the area of the shaded region, (A)

$$= \int_0^1 (x^2 - x^5)\,dx = \left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

Now, $18A = 18 \times \frac{1}{6} = 3$ sq. units

21. We have $x^2 = 4y$ ...(1)

and $x = 4y - 2$ ...(2)

Solving (1) and (2), we get

$(4y - 2)^2 = 4y$

$\Rightarrow 16y^2 + 4 - 16y = 4y \Rightarrow 16y^2 - 20y + 4 = 0$

$\Rightarrow 4y^2 - 5y + 1 = 0 \Rightarrow 4y^2 - 4y - y + 1 = 0$

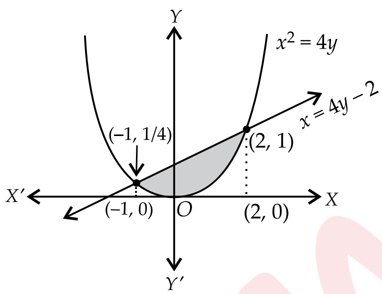
$\Rightarrow 4y(y - 1) - 1(y - 1) = 0 \Rightarrow (4y - 1)(y - 1) = 0$

$\therefore \quad y = 1, \frac{1}{4}$

For $y = 1$, $x = 4 - 2 = 2$ (From (2))

For $y = \frac{1}{4}$, $x = \left(4 \times \frac{1}{4}\right) - 2 = -1$

$\therefore$ The intersecting points are (2, 1) and $\left(-1, \frac{1}{4}\right)$.



Thus the required area

$$= \int_{-1}^{2} \left(\frac{x+2}{4}\right) dx - \int_{-1}^{2} \frac{x^2}{4}\,dx = \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2}$$

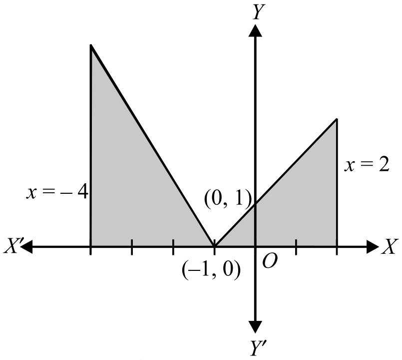
$$= \frac{1}{4}\left[\frac{4}{2} + 4 - \frac{1}{2} + 2\right] - \frac{1}{4}\left[\frac{8}{3} + \frac{1}{3}\right]$$

$$= \frac{1}{4} \cdot \frac{15}{2} - \frac{1}{4} \cdot \frac{9}{3} = \frac{9}{8} \text{ sq. units}$$

22. Here $y = |x + 1| = \begin{cases} x + 1, & \text{if } x \geq -1 \\ -x - 1, & \text{if } x < -1 \end{cases}$

Thus, we get two lines $-x + y = 1$ ...(i), $x + y = -1$...(ii)

Their graphs are as shown and the area to be calculated is shaded.



Hence, the required area

$$= \int_{-1}^{2} (x+1)\,dx + \int_{-4}^{-1} (-x-1)\,dx$$

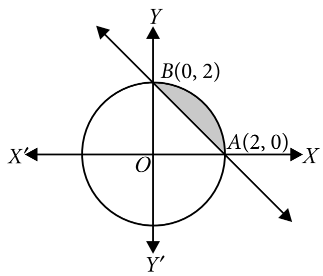
$$= \left[\frac{x^2}{2} + x\right]_{-1}^{2} - \left[\frac{x^2}{2} + x\right]_{-4}^{-1}$$

$$= (2+2) - \left(\frac{1}{2} - 1\right) - \left[\left(\frac{1}{2} - 1\right) - (8 - 4)\right] = \frac{9}{2} + \frac{9}{2}$$

$= 9$ sq. units.

23. The given curves are

$x^2 + y^2 = 4$ ...(i) and $x + y = 2$ ...(ii)



$$\therefore \quad \text{Required area } = \int_0^2 \left[\sqrt{4 - x^2} - (2 - x)\right] dx$$

$$= \left[\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2} - 2x + \frac{x^2}{2}\right]_0^2$$

$$= 0 + 2\sin^{-1}(1) - 4 + 2 - 0 = 2 \cdot \frac{\pi}{2} - 2 = (\pi - 2) \text{ sq.units.}$$