Mid Term

SOLUTIONS

1. **(c) :** Given, $R = \{(x, y) : x, y \in I, x^2 + y^2 \le 4\}$
$= \{(0, 0), (0, -1), (0, 1), (0, -2), ..., (-2, 0)\}$
$\therefore$ Domain of $R = \{x : (x, y) \in R\} = \{-2, -1, 0, 1, 2\}$

2. **(b) :** By equality of two matrices, equating the corresponding elements, we get
$2a + b = 4,\ 5c - d = 11$
$a - 2b = -3,\ 4c + 3d = 24$
Solving these equations, we get
$a = 1, b = 2, c = 3$ and $d = 4$.

3. **(a) :** We have, $y = ax^2 + b \Rightarrow \frac{dy}{dx} = 2ax$
$\left.\frac{dy}{dx}\right|_{x=2} = 2a \times 2 = 4a$

4. $2x^4 - x^2 - 20$

5. Let r be the radius and A be the area of circle.

Given that $\frac{dr}{dt} = 3$ cm/sec ...(i)

We know that, area of circle $A = \pi r^2$

$\therefore \frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi r \cdot 3$ [Using (i)]
$= 6\pi r$

$\therefore \left(\frac{dA}{dt}\right)_{r=2\text{ cm}} = 12\pi \text{ cm}^2/\text{s}$

6. Let $I = \int \frac{1+\cot x}{x + \log \sin x}dx$

Put $x + \log \sin x = t \Rightarrow (1 + \cot x)dx = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + C = \log|x + \log \sin x| + C$

7. $y = \log_7(\log x) = \frac{\log(\log x)}{\log 7}$

$\therefore \frac{dy}{dx} = \frac{1}{\log 7}\cdot\frac{1}{\log x}\cdot\frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{x\log 7\log x}$

8. $A + B + C = O \Rightarrow C = -[A + B]$

$\Rightarrow C = \begin{bmatrix} -3 & 4 & -1 \\ -3 & 0 & -1 \end{bmatrix}$

9. Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 5 & 4 & 1 \\ -2 & 4 & 1 \\ 2 & -6 & 1 \end{vmatrix}$

$= \frac{1}{2}[5(10) - 4(-4) + 1(4)] = \frac{1}{2}\times 70 = 35$

$\therefore$ Area of triangle is 35 sq. units.

10. Let $\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3} = \theta \therefore \cos 2\theta = \frac{\sqrt{5}}{3}$

Now, $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\} = \tan\theta = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$

$= \sqrt{\frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}}} = \sqrt{\frac{(3-\sqrt{5})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})}} = \frac{3-\sqrt{5}}{\sqrt{9-5}} = \frac{3-\sqrt{5}}{2}$

$\therefore \tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right) = \frac{3-\sqrt{5}}{2}$

11. We have, $f(x) = kx^3 - 9kx^2 + 9x + 3$
$\Rightarrow f'(x) = 3kx^2 - 18kx + 9 = 3(kx^2 - 6kx + 3)$
For $f'(x) > 0\ \forall x \in R$
$\Rightarrow 3(kx^2 - 6kx + 3) > 0\ \forall x \in R$
$\Rightarrow kx^2 - 6kx + 3 > 0\ \forall x \in R$
$\Rightarrow k > 0$ and $36k^2 - 12k < 0$
$\Rightarrow k > 0$ and $12k(3k - 1) < 0$
$\Rightarrow k > 0$ and $3k - 1 < 0$ [$\because k > 0$]
$\Rightarrow k > 0$ and $k < \frac{1}{3} \Rightarrow k \in \left(0, \frac{1}{3}\right)$

Hence, $f(x)$ is increasing in R, if $k \in \left(0, \frac{1}{3}\right)$

12. We have, L.H.L. (at $x = 0$)
$= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} m(x^2 - 2x) = m(0 - 0) = 0$
R.H.L. (at $x = 0$) $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cos x = 1$
and $f(0) = \cos 0 = 1$
As, $\lim_{x\to 0^-} f(x) \ne \lim_{x\to 0^+} f(x)$
$\therefore f(x)$ is not continuous at $x = 0$ for any value of m.

13. Let $I = \int_{-5}^{0} f(x)dx$

We have $f(x) = |x| + |x + 3| + |x + 6|$
When $-5 < x < -3$, then
$f(x) = -x - x - 3 + x + 6 = -x + 3$
When $-3 < x < 0$, then $f(x) = -x + x + 3 + x + 6 = x + 9$

$\therefore \quad I = \int_{-5}^{-3} (-x+3)dx + \int_{-3}^{0} (x+9)dx$

$= \left[\frac{-x^2}{2} + 3x\right]_{-5}^{-3} + \left[\frac{x^2}{2} + 9x\right]_{-3}^{0}$

$= \left(\frac{-9}{2} - 9\right) - \left(-\frac{25}{2} - 15\right) + 0 - \left(\frac{9}{2} - 27\right)$

$= \frac{-9}{2} + \frac{25}{2} - \frac{9}{2} - 9 + 15 + 27 = \frac{7}{2} + 33 = \frac{73}{2}$

14. Let the side of the base be x and height be y.
Volume of box $= x^2y = 80$ (Given)

$\Rightarrow y = \frac{80}{x^2}$

Area of base $= x^2$, area of top $= x^2$
and total area of sides $= 4\,xy$

$\therefore$ Cost of making the box

$C(x)$ = cost for the (base + top + sides) + labour charges
$= 3x^2 + 2x^2 + 2\,(4xy) + 200$

$\Rightarrow C(x) = 5x^2 + 2\left(4x\,\frac{80}{x^2}\right) + 200$

$\Rightarrow C(x) = 5x^2 + \frac{640}{x} + 200 \Rightarrow C'(x) = 10x - \frac{640}{x^2}$

Now, $C'(x) = 0$

$\Rightarrow 10x - \frac{640}{x^2} = 0 \Rightarrow 10x^3 = 640$

$\Rightarrow \quad x^3 = 64 \Rightarrow x = 4$

$C''(x) = 10 + \frac{1280}{x^3} > 0$

Thus, cost is minimum when $x = 4$.

$\therefore \quad y = \frac{80}{x^2} = \frac{80}{16} = 5$

Thus dimensions are 4 m, 4 m and 5 m.

15. We have, $y = \frac{1}{\sqrt{3}}x$ ...(i)

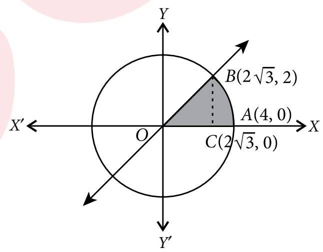
and $x^2 + y^2 = 16$ ...(ii)

Curves (i) and (ii) intersect each other at $(2\sqrt{3}, 2)$ and $(-2\sqrt{3}, -2)$.

$\therefore$ Required area = Area of region $OBAO$
= area ΔOBC + area of region $BCAB$

$= \int_0^{2\sqrt{3}} \frac{x}{\sqrt{3}}dx + \int_{2\sqrt{3}}^{4} \sqrt{16 - x^2}\,dx$

$= \left[\frac{x^2}{2\sqrt{3}}\right]_0^{2\sqrt{3}} + \left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_{2\sqrt{3}}^{4}$



$= 2\sqrt{3} + 8\left(\frac{\pi}{2}\right) - 2\sqrt{3} - \frac{8\pi}{3}$

$= \frac{12\pi - 8\pi}{3} = \frac{4\pi}{3}$ sq.units