Post-Mid Term

SOLUTIONS

1. **(a) :** Here, $\sqrt{(1)^2+(-3)^2+(2)^2}=\sqrt{14}$

$\therefore$ Direction cosines are $\frac{1}{\sqrt{14}},\frac{-3}{\sqrt{14}},\frac{2}{\sqrt{14}}$.

or $\left(\frac{-1}{\sqrt{14}},\frac{3}{\sqrt{14}},\frac{-2}{\sqrt{14}}\right)$.

2. **(c) :** We know that $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2$

$\Rightarrow (25)^2+|\vec{a}\cdot\vec{b}|^2=(5)^2\times(13)^2$

$\Rightarrow \vec{a}\cdot\vec{b}=\sqrt{3600}=\pm 60$

3. **(b) :** Let $I=\int\cos^3 x\, e^{\log\sin x}dx=\int\cos^3 x\sin x\,dx$

Putting $\cos x = t \Rightarrow -\sin x\,dx = dt$, we get

$I=-\int t^3dt=-\frac{t^4}{4}+C=-\frac{\cos^4 x}{4}+C$

4. Linear.

5. Independent.

6. f is not onto as $1 \in Z$, but there does not exists any number x in Z such that $f(x) = 2x = 1$.

7. The sample space S associated to the given random experiment is given by

$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$ and $B = \{(H, 6), (T, 6)\}$ and $A = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6)\}$

$\therefore$ Required probability $= P(B \mid A)$

$\Rightarrow$ Required probability $=\frac{P(A\cap B)}{P(A)}=\frac{\frac{1}{12}}{\frac{6}{12}}=\frac{1}{6}$

8. We have to find the probability that all of them are of same colours, *i.e.*, P(all are white or all are red)

$=P(WWW)+P(RRR)=\frac{7}{15}\times\frac{9}{15}\times\frac{5}{12}+\frac{8}{15}\times\frac{6}{15}\times\frac{7}{12}$

$=\frac{315+336}{2700}=\frac{651}{2700}=\frac{217}{900}$

9. Here, $\det A=\begin{vmatrix}x+1 & -1 & 0\\ 2 & x+4 & 0\\ 0 & 0 & x\end{vmatrix}$

$=(x+1)\begin{vmatrix}x+4 & 0\\ 0 & x\end{vmatrix}-(-1)\begin{vmatrix}2 & 0\\ 0 & x\end{vmatrix}+0$ (Expanding along R_1)

$= (x + 1) \{x(x + 4) - 0\} + (2x - 0)$

$= x (x + 1) (x + 4) + 2x$

$= x \{(x + 1) (x + 4) + 2\} = x (x^2 + 5x + 6)$

Hence, $\det A = 0 \Rightarrow x(x^2+5x+6)=0$

$\Rightarrow x(x+2)(x+3)=0 \Rightarrow x=0,-2,-3$

10. We have, $y = \tan x \Rightarrow \frac{dy}{dx}=\sec^2 x$

$\Rightarrow \frac{d^2y}{dx^2}=2\sec x(\sec x\tan x)=2\sec^2 x\tan x$

$\Rightarrow \frac{d^2y}{dx^2}=2\frac{dy}{dx}\times y=2y\frac{dy}{dx}$

11. We have, $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}-5\hat{k}$

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 3\\ 2 & 3 & -5\end{vmatrix}$

$=(10-9)\hat{i}-(-5-6)\hat{j}+(3+4)\hat{k}=\hat{i}+11\hat{j}+7\hat{k}$

Now, $\vec{a}\cdot(\vec{a}\times\vec{b})=(\hat{i}-2\hat{j}+3\hat{k})\cdot(\hat{i}+11\hat{j}+7\hat{k})$

$= 1 - 22 + 21 = 0$

Therefore, $\vec{a}$ and $\vec{a}\times\vec{b}$ are perpendicular to each other.

12. Let $P = x^2y^3$

It is given that $x + y = 15 \Rightarrow x = 15 - y$

$\therefore P = (15 - y)^2 y^3$

$\Rightarrow \frac{dP}{dy}=3y^2(15-y)^2-2(15-y)y^3$

$= (15 - y) y^2 [3(15 - y) - 2y]$

$= y^2(15 - y) (45 - 3y - 2y) = 5y^2 (15 - y) (9 - y)$

Now, $\frac{dP}{dy}=0$

$\Rightarrow 5y^2(15 - y)(9 - y) = 0 \Rightarrow y = 0, y = 15, y = 9$

{$y = 0$ and 15 not possible as $0 < y < 15$}

$\therefore y = 9$

Also, $\frac{d^2P}{dy^2}=5[(15-y)(9-y)2y-y^2(9-y)-y^2(15-y)]$

$\therefore \left(\frac{d^2P}{dy^2}\right)_{y=9}=5(0-0-9^2\times 6)=-2430<0$

So, P has maximum value at $y = 9$.
Thus, the numbers are $x = 15 - 9 = 6$ and $y = 9$.

13. We have, $(x+2)\frac{dy}{dx} = x^2 + 4x - 9 \Rightarrow \frac{dy}{dx} = \frac{x^2+4x-9}{x+2}$ $[x \neq -2]$

$$\Rightarrow \quad dy = \left(\frac{x^2+4x-9}{x+2}\right)dx$$

Integrating both sides, we get $\int dy = \int \frac{x^2+4x-9}{x+2}dx$

$$\Rightarrow \quad \int dy = \int\left(x+2-\frac{13}{x+2}\right)dx$$

$$\Rightarrow \quad y = \frac{x^2}{2} + 2x - 13\log|x+2| + C,$$

which is the solution of the given differential equation.

14. Let the direction ratios of the required line be a, b, c.
Since it is perpendicular to the two given lines,

$$\frac{x-8}{8} = \frac{y+9}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$$

$\therefore \quad 8a - 16b + 7c = 0$ ...(i)

and $3a + 8b - 5c = 0$ ...(ii)

Solving (i) and (ii) by cross-multiplication, we get

$$\frac{a}{80-56} = \frac{b}{21+40} = \frac{c}{64+48} = \lambda \Rightarrow \frac{a}{24} = \frac{b}{61} = \frac{c}{112} = \lambda$$

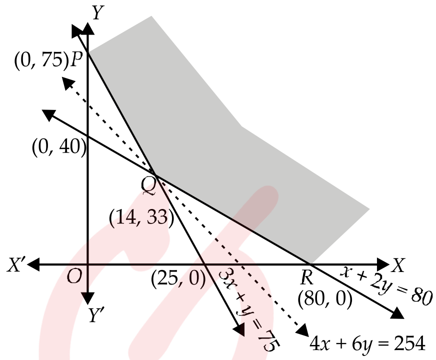
$\Rightarrow \quad a = 24\lambda, b = 61\lambda, c = 112\lambda$

So, the direction ratios of the required line are 24, 61, 112. The equation of required line which passes through (1, 2, –4) and whose direction ratios are 24, 61,112 is

$$\frac{x-1}{24} = \frac{y-2}{61} = \frac{z+4}{112}.$$

15. Minimise $Z = 4x + 6y$,
subject to the constraints: $x \geq 0$, $y \geq 0$, $x + 2y \geq 80$ and $3x + y \geq 75$.

The feasible region is shown in the figure and it is unbounded.



Corner points	Minimise $Z = 4x + 6y$
$P(0, 75)$	$4 \times 0 + 75 \times 6 = 450$
$Q(14, 33)$	$14 \times 4 + 6 \times 33 = 254$ (Minimum)
$R(80, 0)$	$80 \times 4 + 0 \times 6 = 320$

Since, feasible region is unbounded and open half plane determined by $4x + 6y \leq 254$ has no points in common with the feasible region.

$\therefore \quad Z$ is minimum at $Q(14, 33)$ and minimum value of $Z = 254$