Systems of Particles and Rotational Motion

CHAPTER 7

EXAM DRILL

ANSWERS

1. **(d) :** $\vec{\tau} = \vec{r} \times \vec{F}$

Vector $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$.

$\therefore \quad \vec{r} \cdot \vec{\tau} = 0$ and $\vec{F} \cdot \vec{\tau} = 0$

2. **(d) :** Torque due to the central force is zero.

$\vec{\tau} = \frac{d\vec{L}}{dt} = 0 \quad \Rightarrow \quad \vec{L} = \text{constant}.$

3. **(b) :** A circular wire behaves like a ring.

By perpendicular axes theorem, $I_D + I_D = \frac{1}{2}MR^2$

$\therefore \quad I_D = \frac{1}{4}MR^2$

4. **(d) :** Out of the four given bodies, the centre of mass of a bangle lies outside it whereas in all other three bodies it lies within the body.

5. **(b) :** As the radius increases, the M.I of the sphere increases. As no external torque acts in free space, the speed of rotation decreases but the angular momentum remains constant.

6. **(b) :** By conservation of angular momentum,

$(m + 2M)R^2\omega' = mR^2\omega$

$\omega' = \frac{m\omega}{m + 2M}$

7. **(a) :** The acceleration is given by, $\dfrac{g\sin\theta}{1 + \dfrac{I}{MR^2}}$

8. **(c) :** The body will experience both linear and angular acceleration.

9. **(a) :** For horizontal part of length, L

$x_1 = \frac{L}{2}, y_1 = 0, \; m_1 = KL \quad (\because m \propto L)$

For vertical part of length $2L$,

$x_2 = 0,\; y_2 = L,\; ,m_2 = K \times 2L$

$$X_{COM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{KL\left(\frac{L}{2}\right) + 2KL(0)}{KL + 2KL} = \frac{L}{6} = \frac{1.2}{6} = 0.2 \text{ m}$$

$$Y_{COM} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2} = \frac{KL(0) + 2KL(L)}{KL + 2KL} = \frac{2}{3}L = \frac{2}{3}(1.2) = 0.8 \text{ m}$$

$\therefore \quad \vec{r}_{COM} = (0.2\hat{i} + 0.8\hat{j})\text{ m}$

10. **(b) :** For a disc rolling down an incline plane,

$l = \frac{1}{3}g\sin\alpha$

11. **(a)**

12. **(i) (d) :** Energy is conserved for all the three bodies. So, final kinetic energy for all will be same.

(ii) **(a) :** For ring, $\frac{k^2}{R^2} = 1$

For disc, $\frac{k^2}{R^2} = \frac{1}{2}$

For sphere, $\frac{k^2}{R^2} = \frac{2}{5} = 0.4$

$$a = \frac{g\sin\theta}{1 + \dfrac{k^2}{R^2}}$$

More is the acceleration, smaller will be the time taken.

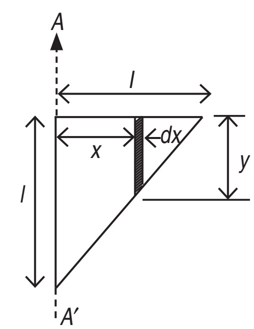
$v_{sphere} > v_{disc} > v_{ring}$

(iii) **(c) :** The frictional force for ring is maximum.

13. Angular velocity of spinning earth $= \frac{2\pi}{86400}\text{ rad s}^{-2}$

14. Consider an element of thickness dx at a distance x from AA'. Its height is $y = l - x$.

Mass of the element,



$dm = \frac{M}{(l^2/2)}(l - x)dx$

$= \frac{2M}{l^2}(l - x)dx$

$(d')_{AA'} = \frac{2M}{l^2}(l - x)dx \times x^2$

$$I_{AA'} = \frac{2M}{l^2}\int_0^1 (lx^2 - x^2)dx = \frac{2M}{l^2}\left[\frac{l^4}{3} - \frac{l^4}{4}\right] = \frac{ML^2}{6}$$

15. The centre of mass of a rectangle lies at the point of intersection of its diagonal.

16. No. A torque produces angular acceleration.

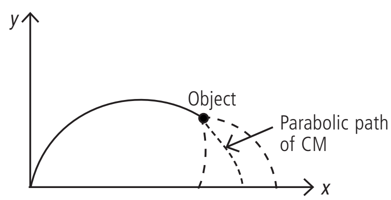
OR

The torque applied on the nut by the wrench is equal to the force multiplied by the perpendicular distance from the axis of rotation. Hence to increase torque, a wrench of longer arm is preferred.

17. $\tau = rF\sin\theta$, so torque is zero if (i) r is zero (ii) F is zero and (iii) $\theta = 0°$ or $180°$.

18. Motion of CM of fire crackers exploding in air :

Suppose a fire crackerfired from O explodes into fragments in mid-air at P. The forces leading to explosion were internal forces. They contribute nothing to the motion of centre of mass. The total external force, namely the force of gravity acting on the cracker is the same before and after explosion. The centre of mass of all the fragments will therefore, continues to move along the same parabolic trajectory as it could have followed, if there were no explosion.



19. For solid cylinder, $a = \frac{2}{3}g\sin\theta$

$\therefore \quad a = \frac{2}{3}g\sin 30° = \frac{g}{3}$

OR

Radial component of the applied force cannot produce torque.

$\tau = rF\sin\theta = 0 \times F\sin\theta = 0$

20. Suppose an external force $\vec{F}_{CM}$ acts on a system of mass M and produces an acceleration $\vec{a}_{CM}$ at its centre of mass. Then,

$\vec{F}_{tot} = M\vec{a}_{CM}$

In the absence of any external force, $\vec{F}_{CM} = 0$,

$\therefore \quad M\vec{a}_{CM} = 0 \Rightarrow \vec{a}_{CM} = 0$ or $\frac{d\vec{v}_{CM}}{dt} = 0$

As the derivative of a constant is zero, so

$\vec{v}_{CM} = \text{constant}$

where $\vec{v}_{CM}$ is the velocity of the centre of mass. Hence in the absence of any external force, the centre of mass of a system moves with a uniform velocity.

21. (i) Since $\alpha = 10$ rad/s^2 (constant)

$\therefore$ We can use, $\omega = \omega_0 + \alpha t$

where $\omega_0 = 0$ (Initial angular velocity).

So, $\omega = 0 + 10(5) = 50$ rad/s

(ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

Angular displacement, $\theta = 0 + \frac{1}{2}(10)(5)^2 = 125 \text{ rad}$,

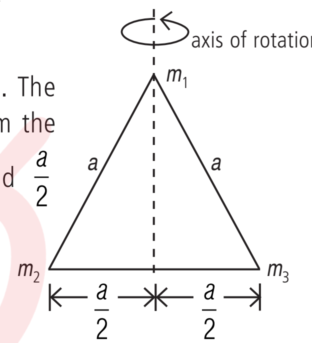
We know that, $\theta = (2\pi)n$,

where n is the number of revolution and 2π is the angle turned in one revolution.

$\therefore \quad n = \frac{125}{2\pi} = \frac{125}{2(3.14)} = 19.9 \Rightarrow n = 20$ revolution

OR

As shown in figure, the axis of rotation passes through m_1. The distances of m_1, m_2 and m_3 from the axis of rotation are 0, $\frac{a}{2}$ and $\frac{a}{2}$ respectively.



$\therefore$ M.I of the system about the altitude through m_1 is,

$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$

$I = m_1(0)^2 + m_2\left(\frac{a}{2}\right)^2 + m_3\left(\frac{a}{2}\right)^2$ or $I = \frac{a^2}{4}(m_2 + m_3)$

22. (i) A heavy wheel called flywheel, is attached to the shaft of steam engine, automobile engine etc. Because of its large M.I, the flywheel opposes the sudden increase or decrease of the speed of the vehicle. It allows a gradual change in speed and prevents jerky motions.

(ii) In a bicycle, bullock-cart etc., the moment of inertia is increased by concentrating most of the mass at the rim of the wheel and connecting the rim to the axle through the spokes. Even after we stop paddling, the wheels of a bicycle continue to rotate for some time due to their large moment of inertia.

23. During collapse the total angular momentum of an isolated star is conserved. Hence,

$I_1\omega_1 = I_2\omega_2$

or $\frac{2}{5}MR_1^2\omega_1 = \frac{2}{5}MR_2^2\omega_2$ or $R_1^2\omega_1 = R_2^2\omega_2$

$\therefore \quad \omega_2 = \frac{R_1^2}{R_2^2}\omega_1$

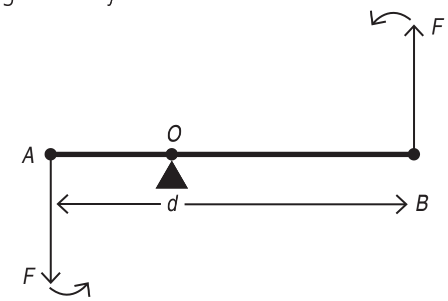
But $R_1 = 10^6$ km, $R_2 = 10^4$ km and $\omega_1 = 10^{-6}$ rad s^{-1}

$\therefore \quad \omega_2 = \frac{(10^6)^2}{(10^4)^2} \times 10^{-6} = 0.01 \text{ rad s}^{-1}$

24. A pair of equal and opposite forces acting on a body along two different lines of action constitiute a couple. A couple has a turning effect, but no resultant force acts on a body.

The moment of couple can be found by taking the moments of the two forces about any point and then adding them.

In figure, two opposite forces, each of magnitude 'F' act at two points A and B of a rigid body, which can rotate about point 'O'. The turning tendency of the two forces is anti-clockwise.



Moment or torque of the couple about 'O' is

$\tau = F \times AO + F \times OB = F(AO + OB) = F \times AB$ or $\tau = Fd$

Moment of a couple = Force × perpendicular distance between two forces.

Hence, the moment of a couple is equal to the product of either of the two forces and the perpendicular distance called the arm of the couple between their line of action. Note that the torque exerted by couple about O does not depend on the position of O. Hence, torque or moment of a couple is independent of the choice of the fulcrum or the point of rotation.

25. As we know, $x_{CM} = \frac{1}{M}\int_0^L x\,dm = \frac{1}{M}\int_0^L (x)\lambda dx$

$$x_{CM} = \frac{1}{M}\int_0^L x(\rho x)dx = \frac{\rho}{M}\int_0^L x^2 dx = \frac{\rho}{M} \times \frac{L^3}{3} = \frac{\rho L^3}{3M}$$

$$\text{But } M = \int dm = \int_0^L \lambda dx = \int_0^L \rho x dx = \frac{\rho L^2}{2}$$

Thus, $x_{CM} = \dfrac{\rho L^3}{3\left(\dfrac{\rho L^2}{2}\right)} = \dfrac{2}{3}L$ (from origin $x = 0$)

OR

Here, $m_1 = 2$ kg ; $m_2 = 10$ kg

Length of rod = 1.2 m

Suppose the centre of mass lies at distance 'x' from mass m_1.

Then, $m_1 x = m_2(1.2 - x)$

$2x = 10 \times (1.2 - x)$

$x = 1$ m

As the rod is weightless, its moment of inertia about any axis is zero.

Moment of inertia of m_1 about CM

$= m_1 x^2 = 2(1)^2 = 2$ kg m^2

M.I of m_2 about CM $= m_2(1.2 - x)^2 = 10(1.2 - 1)^2$

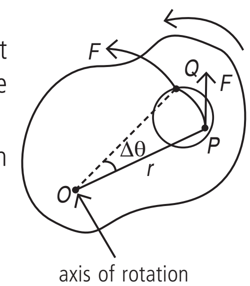
$= 0.4$ kg m^2

Total M.I = 2 + 0.4 = 2.4 kg m^2.

26. Internal elastic forces in a wheel is responsible for the centripetal acceleration of its particles. These forces are in pairs and cancel each other because they are part of a symmetrical system.

For half-wheel, the CM does not coincide with its centre so mass distribution is not symmetrical about the axis of rotation. Therefore, the direction of angular momentum does not coincide with the direction of angular velocity and hence to maintain rotation, an external torque is required.

27. As shown in the figure, suppose a body undergoes an angular displacement $\Delta\theta$ under the action of a tangential force F.



The work done in the rotational motion of the body

$\Delta W = F \times$ distance along the arc PQ

But $\Delta\theta = \dfrac{\text{Arc}}{\text{Radius}} = \dfrac{\text{Arc } PQ}{r}$

$\therefore \quad PQ = r\Delta\theta$

Hence $\Delta W = F \times r\Delta\theta$ or $\Delta W = \tau\Delta\theta$

i.e., work done by a torque = torque × angular displacement

If the torque applied is not constant, but variable, the total work done by the torque is given by

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta$$

We know that, $\Delta W = \tau\Delta\theta$

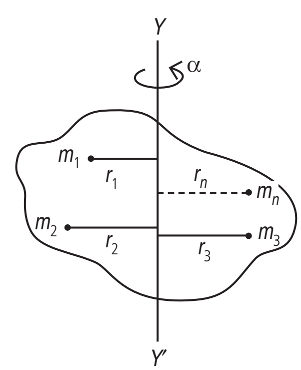
Dividing both sides by Δt, we get

$$\frac{\Delta W}{\Delta t} = \tau\frac{\Delta\theta}{\Delta t} \quad \text{or} \quad P = \tau\omega$$

i.e., power = torque × angular velocity

28. Consider a rigid body rotating about a given axis with uniform acceleration α, under the action of a torque.

Let the body consists of n particles of masses $m_1, m_2, m_3, \ldots m_n$ at perpendicular distance $r_1, r_2, r_3, \ldots, r_n$ respectively from the axis of rotation.



When a torque acts on a body capable of rotation about an axis, it produces an angular acceleration in the body. If the angular velocity of each particle is ω then the angular acceleration, $\alpha = d\omega/dt$ will be same for all particles of the body. The linear acceleration will depend on their distances $r_1, r_2, r_3, \ldots, r_n$ from the axis of rotation.

If $a_1, a_2, a_3, \ldots a_n$ are acceleration of the particles then,

$a_1 = r_1\alpha,\ a_2 = r_2\alpha,\ a_3 = r_3\alpha, \ldots, a_n = r_n\alpha$

Force on particle of mass $m_1, F_1 = m_1 r_1 \alpha$

Moment of force F_1 about the axis of rotation, $\tau_1 = F_1 r_1 = m_1 r_1^2\alpha$

Similarly, moments of forces on other particles about the axis of rotation are $m_2r_2^2\alpha, m_3r_3^2\alpha, ..., m_nr_n^2\alpha$.

$\therefore$ Total torque acting on the rigid body,

$\tau = \tau_1 + \tau_2 + \tau_3 + ... + \tau_n$

or $\tau = m_1r_1^2\alpha + m_2r_2^2\alpha + m_3r_3^2\alpha + ... + m_nr_n^2\alpha$
$= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ... + m_nr_n^2)\,\alpha$

or $\tau = \left(\sum_{i=1}^{n} m_i r_i^2\right)\alpha$

But $\sum_{i=1}^{n} m_i r_i^2 = I$, moment of inertia of the body about the given axis

$\therefore$ $\tau = I\alpha$

Torque = moment of inertia × angular acceleration.

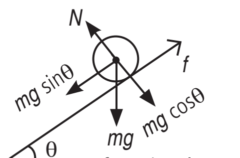
OR

Suppose the radius of the sphere is r. The forces acting on the sphere are shown in figure. They are

(i) weight mg

(ii) normal force N and

(iii) friction, f



Let the linear acceleration of the sphere down the plane be a. The equation for the linear motion of the centre of mass is

$$mg\sin\theta - f = ma \quad ...\text{(i)}$$

As the sphere rolls without slipping, its angular acceleration about the centre is a/r. Thus, the equation of rotational motion about the centre of mass is

$$fr = \left(\frac{2}{5}mr^2\right)\left(\frac{a}{r}\right) \Rightarrow f = \frac{2}{5}ma \quad .... \text{(ii)}$$

From eqn. (i) and (ii),

$$a = \frac{5}{7}g\sin\theta \quad \text{and} \quad f = \frac{2}{7}mg\sin\theta$$

The normal force is equal to $mg\cos\theta$ as there is no acceleration perpendicular to the incline. The maximum friction that can act is, $\mu mg\cos\theta$, where μ is the coefficient of static friction.

for pure rolling,

$$\mu mg\cos\theta > \frac{2}{7}mg\sin\theta \quad \text{or} \quad \mu > \frac{2}{7}\tan\theta$$

29. (a) Disc is rotating only about its horizontal axis before being brought in contact with the table. Hence, its CM is at rest; $v_{CM} = 0$

(b) The linear velocity of a point on the rim of disc decreases due to friction between disc and table.

(c) Linear speed of the CM of disc increases when disc is placed in contact with the table, because its acceleration becomes $a_{CM} = \mu_k g$.

(d) Force of friction between table and disc.

(e) Condition for rolling to begin, $v_{CM} = R\omega$.

(f) Force of friction on disc, $F = \mu_k mg$.

Acceleration produced in centre of mass due to friction

$$a_{CM} = \frac{F}{m} = \frac{\mu_k mg}{m} = \mu_k g \quad ...\text{(i)}$$

Using, $v = u + at$

$\Rightarrow$ $v_{CM} = 0 + \mu_k gt = \mu_k gt$ ...(ii)

Angular acceleration is produced by torque due to friction,

$$\alpha = \frac{\tau}{I} = \frac{(\mu_k mg)R}{I}$$

Now, $\omega = \omega_0 + \alpha t$

$$\Rightarrow \omega = \omega_0 + \frac{\mu_k mgR}{I}t$$

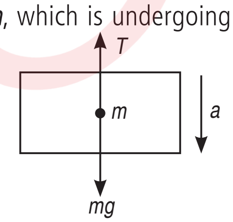
For pure rolling, $v_{CM} = \omega R$

$$\Rightarrow \mu_k gt = \left(\omega_0 + \frac{\mu_k mgR}{I}t\right)R$$

$$\Rightarrow \mu_k gt = R\omega_0 + \frac{\mu_k mgR^2}{I}t \Rightarrow t = \frac{R\omega_0}{\mu_k g\left(1 - \frac{mR^2}{I}\right)}$$

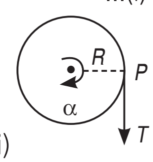
OR

(a) For the block m, which is undergoing translation.



$mg - T = ma$ ...(i)

For the pulley, which undergoes pure rotation,



Moment of inertia = I

$\tau = T \cdot R$

As $\tau = I\alpha \Rightarrow T\cdot R = I\alpha$ ...(ii)

Acceleration of point 'P' $= \alpha R$ as it is on the periphery of the pulley. But a_P = acceleration of block as they are interconnected by a unstretchable string.

Thus, $a = \alpha R$

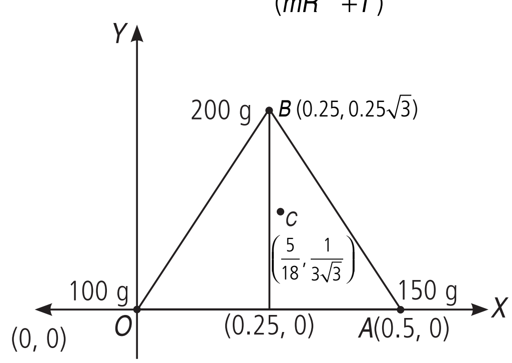
Equation (i) becomes $mg - T = m(\alpha R)$

From (ii) $T = \dfrac{I\alpha}{R}$

Thus, $mg - \dfrac{I\alpha}{R} = m\alpha R$

$mgR = mR^2\alpha + I\alpha$ or $\alpha = \dfrac{mgR}{(mR^2 + I)}$

(b)



With the X–and Y–axes chosen as shown in figure, the co-ordinates of points O, A and B forming the equilateral triangle are respectively (0, 0), (0.5, 0), (0.25, 0.25$\sqrt{3}$). Let the masses 100 g, 150 g and 200 g be located at O, A and B respectively. Then,

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{100(0) + 150(0.5) + 200(0.25)\text{ gm}}{(100 + 150 + 200)\text{ g}}$$

$$= \frac{75 + 50}{450}\text{m} = \frac{125}{450}\text{m} = \frac{5}{18}\text{m}$$

$$Y = \frac{100(0) + 150(0) + 200(0.25\sqrt{3})\text{ gm}}{450\text{ g}}$$

$$= \frac{50\sqrt{3}}{450}\text{ m} = \frac{\sqrt{3}}{9}\text{ m} = \frac{1}{3\sqrt{3}}\text{ m}$$

30. (a) Given, area of square = area of rectangle

$\Rightarrow$ $c^2 = ab$

Also, $a > b$

(i) $\frac{I_{xR}}{I_{xS}} = \frac{Mb^2/12}{Mc^2/12} = \frac{b^2}{c^2} = \frac{b^2}{ab} = \frac{b}{a} < 1$

(ii) $\frac{I_{yR}}{I_{yS}} = \frac{Ma^2/12}{Mc^2/12} = \frac{a^2}{c^2} = \frac{a^2}{ab} = \frac{a}{b} > 1$

(iii) $I_{zR} = \frac{1}{12}M(a^2 + b^2)$; $I_{zS} = \frac{1}{12}M(c^2 + c^2)$

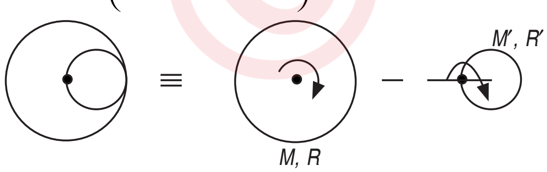
Now, $I_{zR} - I_{zS} = \frac{1}{12}M[a^2 + b^2 - 2c^2]$

$$= \frac{1}{12}M[a^2 + b^2 - 2ab] \qquad [\because c^2 = ab]$$

$\Rightarrow$ $I_{zR} - I_{zS} = \frac{1}{12}M(a - b)^2 > 0$

$\therefore$ $I_{zR} > I_{zS}$ $\Rightarrow$ $\frac{I_{zR}}{I_{zS}} > 1$

(b) $I = \frac{MR^2}{2} - \left(\frac{M'R'^2}{2} + M'(R')^2\right)$



where $R' = \frac{R}{2}$ and $M' = \frac{M}{4}$

On putting the values we get, $I = \frac{13MR^2}{32}$

(c) For the rod of mass M and length l with $a = l/2$, we get

$$I = \frac{Ml}{12}$$

Using the parallel axes theorem, $I' = I + Ma^2$

$$I' = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$$

OR

(a) Here, $m = 5$ kg ; $r = 30$ cm $= 0.3$ m

$\omega_1 = 0$, $\omega_2 = ?$; $t = 30$ s

Angular impulse = change in angular momentum

$$3 = I(\omega_2 - \omega_1) = \frac{1}{2}mr^2(\omega_2 - \omega_1)$$

$$= \frac{1}{2} \times 5 \times \left(\frac{3}{10}\right)^2(\omega_2 - 0)$$

$$\omega_2 = \frac{3 \times 2 \times 100}{5 \times 9} = \frac{40}{3}$$

As $\omega_2 = \omega_1 + \alpha t$

$\therefore$ $\frac{40}{3} = 0 + \alpha \times 4$ or $\alpha = \frac{10}{3}\text{ rad s}^{-2}$

As impulse is imparted after every 4 seconds, impulses imparted are at $t = 0, 4, 8, 12, 16, 20, 24, 28$ etc.

The last one lasts upto $t = 32$ sec. (before the next impulse is imparted)

As $\omega_2 = \omega_1 + \alpha t$

$\therefore$ $\omega_2 = 0 + \frac{10}{3} \times 32 = \frac{320}{3} = 106.7\text{ rad s}^{-1}$

(b) Torque is not produced by weight about y-axis.

A force can produce torque only along direction normal to itself because $\vec{\tau} = \vec{r} \times \vec{F}$.

So, when the door is in the xy-plane, the torque produced by its weight can only be along $+z$ or $-z$ direction.